APPM 1235 Exam 3 version A

1. [5 points each, 25 points total] Find the exact value of each of the following. If an answer does not exist, write “DNE.”

(a) \( \sin^{-1} \left( \sin \frac{11\pi}{6} \right) \)

(b) \( \sec \left( \cot^{-1} \frac{15}{8} \right) \)

(c) \( \cos \frac{7\pi}{12} \)

(d) \( \cos \left( 2 \tan^{-1} \frac{12}{5} \right) \)

(e) \( \sin \left( \cos^{-1} \frac{1}{2} + \tan^{-1} 1 \right) \)

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(a) \( \sin^{-1} \left( \sin \frac{11\pi}{6} \right) = \sin^{-1} \left( -\frac{\sqrt{3}}{2} \right) = -\frac{\pi}{3} \)

(b) \( \sec \left( \cot^{-1} \frac{15}{8} \right) = \frac{17}{15} \)

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(c) \( \cos \left( \frac{7\pi}{12} \right) = \cos \left( \frac{\pi}{3} + \frac{\pi}{4} \right) = \cos \frac{\pi}{3} \cos \frac{\pi}{4} - \sin \frac{\pi}{3} \sin \frac{\pi}{4} = \frac{\sqrt{6} - \sqrt{2}}{4} \)

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(d) \( \cos \left( 2 \tan^{-1} \frac{12}{5} \right) \)

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(e) \( \sin \left( \cos^{-1} \frac{1}{2} + \tan^{-1} 1 \right) \)

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2. [11 points] As the moon revolves around the earth, the side that faces the earth is usually just partially illuminated by the sun. The phases of the moon describe how much of the surface appears to be in sunlight. An astronomical measure of phase is given by the fraction $F$ of the lunar disk that is lit. When the angle between the sun, earth and moon is $\theta$, where $0 \leq \theta \leq 360^\circ$, then

$$F = \frac{1}{2} (1 - \cos \theta)$$

Determine the angles $\theta$ that correspond to the following phases.

(a) $F = 0.25$ (crescent moon)
(b) $F = 1$ (full moon)
(c) What is the range of values that $F$ can take?

\[ \begin{align*}
\text{a)} & \quad \frac{1}{4} = \frac{1}{2} (1 - \cos \theta) \Rightarrow \cos \theta = \frac{1}{2} \\
& \quad \theta = 60^\circ, 300^\circ \\
\text{b)} & \quad 1 = \frac{1}{2} (1 - \cos \theta) \Rightarrow \cos \theta = -1 \\
& \quad \theta = 180^\circ \\
\text{c)} & \quad -1 \leq \cos \theta \leq 1 \\
& \quad 0 \leq 1 - \cos \theta \leq 2 \\
& \quad 0 \leq \frac{1}{2} (1 - \cos \theta) \leq 1
\end{align*} \]
3. [12 points] (a) Find all solutions of \( \sin 2\theta = \cos \theta \).

(b) Find all solutions of \( 2 \cos 2\theta = \sqrt{3} \) on \([0, 2\pi]\).

\( a) \quad 2\sin \theta \cos \theta - \cos \theta = 0 \)
\[ \cos \theta (2\sin \theta - 1) = 0 \]
\[ \cos \theta = 0 \quad \sin \theta = \frac{1}{2} \]
\[ \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots \]

\( b) \quad \cos 2\theta = \frac{1}{2} \sqrt{3} \)
\[ 2\theta = \frac{\pi}{6} + 2k\pi \]
\[ \theta = \frac{\pi}{12} + k\pi \]
\[ 2\theta = \frac{13\pi}{6} + 2k\pi \]
\[ \theta = \frac{13\pi}{12} + k\pi \]

\( k \) is an integer.
4. [12 points] (a) State 3 consecutive asymptotes of the function \( f(x) = \csc(2x - \pi) \).

(b) State 3 consecutive zeros of \( g(x) = \tan(2x - 1) \).

\[ \text{a) } f(x) = \csc 2(x - \frac{\pi}{2}) \]

\[ P = \frac{2\pi}{2} = \pi \quad P.S. = \frac{\pi}{2} \]

V.A. \( x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots \)

\[ \text{b) } g(x) = \tan(2x - 1) \]

\[ P = \frac{\pi}{2} \]

\[ \text{zero} \quad 0 = 2x - 1 \]

\[ x = \frac{1}{2} \]

\[ \text{zeroes} \quad x = \frac{1}{2}, \frac{1}{2} + \frac{\pi}{2}, \frac{1}{2} + \pi, \ldots \]
5. [20 points] Consider the figure below. Both circles have a radius of 3. Points O are the centers of the circles.

(a) Describe your strategy in words to find area $A_2 + A_3$ (area of regions 2 and 3). Use phrases such as "area of sector" and "area of triangle," etc.

(b) Find the area $A_2 + A_3$. List all formulas used to find the area.

(c) Describe your strategy in words to find the perimeter of region 4.

(d) Find the perimeter of region 4.

\[
\begin{align*}
\text{a)} & \quad A_2 + A_3 = 2 \left( \text{Area of sector} - \text{Area of triangle} \right) \\
\text{b)} & \quad A_2 + A_3 = 2 \left( \frac{1}{2} \cdot (3)^2 \cdot \frac{\pi}{2} - \frac{1}{2} \cdot 3 \cdot 3 \cdot \sin \frac{\pi}{2} \right) \\
& \quad = 2 \left( \frac{9\pi}{4} - \frac{9}{2} \right) = \boxed{\frac{9\pi}{2} - 9} \\
\text{c)} & \quad P = 2 \cdot \text{legs of triangle} + \text{arc length} \\
\text{d)} & \quad P = 3 + 3 + 3 \cdot \frac{\pi}{2} = \boxed{6 + \frac{3\pi}{2}}
\end{align*}
\]
6. [5 points each, 20 points total] The following questions are not related.

(a) The expression \( \frac{1 - \cos x}{\sin x} + \frac{\sin x}{1 - \cos x} \) simplifies to which of the following?

   A. \( \frac{2\sin x}{1 - \cos x} \)  
   B. \( 2\csc x \)  
   C. \( 2 - \tan x \)  
   D. \( 2 - 2\cos x \)

(b) The expression \( \cos^4 x - \sin^4 x \) simplifies to which of the following?

   A. \( \sin 2x \)  
   B. \( (\cos x - \sin x)^4 \)  
   C. \( -1 \)  
   D. \( \cos 2x \)

(c) A pulley wheel turning at a rate of 120 revolutions per minute drives a belt moving at 2 feet per second. What is the radius of the pulley wheel?

   A. \( 2\pi \) feet  
   B. 1 foot  
   C. 5 inches  
   D. \( \frac{1}{2\pi} \) feet  
   E. \( \frac{30}{\pi} \) feet

(d) Rewrite \( \tan \left( \sin^{-1} \frac{2}{\sqrt{x^2 + 2}} \right) \) as an algebraic expression in \( x \).

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\begin{align*}
\text{a)} & \quad \frac{1 - \cos x}{\sin x} \cdot \frac{(1 - \cos x)}{(1 - \cos x)} + \frac{\sin x}{1 - \cos x} \cdot \frac{\sin x}{\sin x} \\
& = \frac{1 - \cos x + \cos^2 x + \sin^2 x}{\sin x(1 - \cos x)} \\
& = \frac{2 - \cos x}{\sin x(1 - \cos x)} = \frac{2\csc x}{\sin x(1 - \cos x)} \\
\text{b)} & \quad \cos^4 x - \sin^4 x = (\cos^3 x - \sin^3 x)(\cos x + \sin x) = \cos x \\
\text{c)} & \quad 120 \text{ rev/min} \cdot 1 \text{ min} \cdot \frac{1 \text{ rad}}{1 \text{ rev}} = 2\pi \text{ rad/min} \\
& \quad 4\pi \text{ rad/sec} = \frac{2\pi}{\sec} \\
\text{d)} & \quad (\sqrt{x^2 + 2})^2 = x^2 + y^2 \\
& \quad x^2 + 2 - 4 = y^2 \\
& \quad y = \sqrt{y^2 - 2} \\
& \quad \tan(\sin^{-1} \frac{2}{\sqrt{x^2 + 2}}) = \frac{2}{\sqrt{x^2 - 2}} \quad \text{(a triangle with side lengths labeled)}