Applied Analysis Preliminary Exam
9:00-12:00 August 16, 2022

Instructions: You have three hours to complete this exam. Work all five problems; each is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are being asked to prove such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name.

Problem 1: Metric Spaces

(a) Suppose that (X, d_X) and (Y, d_Y) are metric spaces. Show that $X \times Y$ is a metric space, with an appropriate metric.
(b) Let $A = \{(\xi_1, \xi_2, \ldots) : \xi_j \in \mathbb{R}\}$ be the space of real sequences and $\mu_j > 0$. Show that

$$d(\xi, \eta) = \sum_{j=1}^{\infty} \mu_j \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|}$$

is a metric on A whenever $\sum_{j=1}^{\infty} \mu_j$ converges.
(c) Let $M \subset \ell^\infty$ be the subspace consisting of all sequences (ξ_j) with at most finitely many nonzero terms. Show that M is not complete.

Solution/Hint:

(a) Let $Z = X \times Y$ and $d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2)$ We need only show that the new metric satisfy the three conditions:
  (1) (symmetry) This is clear since both d_X and d_Y are symmetric
  (2) (positivity) Since $d_X(x_1, x_2) \geq 0$ and $d_Y(y_1, y_2) \geq 0$, then $d(z_1, z_2) \geq 0$. Moreover the only way this can be zero is if both d_X and d_Y vanish which only happens if $x_1 = x_2$ and $y_2 = y_2$, thus we have $z_1 = z_2$.
  (3) (triangle inequality) Let $z_3 = (x_3, y_3) \in Z$, then by the triangle for d_X and d_Y

$$\begin{aligned} d(z_1, z_2) &\leq d_X(x_1, x_3) + d_X(x_3, x_2) + d_Y(y_1, y_3) + d_Y(y_3, y_2) \\ &\leq (d_X(x_1, x_3) + d_Y(y_1, y_3)) + (d_X(x_3, x_2) + d_Y(y_3, y_2)) = d(z_1, z_3) + d(z_3, z_2) \end{aligned}$$

  which is the triangle inequality for d.

  Thus (Z, d) is a metric space.
(b) Note that $d(\xi, \eta)$ is symmetric and positive definite. Note that the function $f(t) = t/(1+t)$ is monotone increasing on $\mathbb{R}^+$ (e.g., $f'(t) > 0$), and $f(t) < 1$. Thus convergence of the μ series implies that $d(\xi, \eta) < \infty$. To show the triangle inequality, write $a = |\xi_j - \eta_j| \leq |\xi_j - \zeta_j| + |\zeta_j - \eta_j| = b + c$, then since $a, b, c \geq 0$, $f(a) \leq f(b + c)$ so

$$\frac{a}{1+a} \leq \frac{b+c}{1+b+c} = \frac{b}{1+b+c} + \frac{c}{1+b+c} \leq \frac{b}{1+b} + \frac{c}{1+c}$$

  This is just what we need for the Triangle inequality, since then

$$\frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} \leq \frac{|\xi_j - \zeta_j|}{1 + |\xi_j - \zeta_j|} + \frac{|\zeta_j - \eta_j|}{1 + |\zeta_j - \eta_j|}$$

  Multiplication by μ_j and summing gives the result.
(c) The sequence $x_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, 0, 0, \ldots) \in M$ for each $n \in \mathbb{N}$. Moreover this is a Cauchy sequence, since

$$d(x_m, x_n) = \frac{1}{n+1}$$

  whenever $m \geq n$. However, $\lim_{n\to\infty} x_n \notin M$. Thus M is not complete.

Problem 2: Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = \frac{x^n y^m}{x^2 + y^2}$$

for $(x, y) \neq (0, 0)$ and $m, n \in \mathbb{N}$. Show that $\lim_{(x,y)\to(0,0)} f(x, y)$ exists if and only if $m + n > 2$.

Solution/Hint: Suppose $(x_k, y_k) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ is a sequence that converges to zero. If $m + n > 2$, then at least one of the values must be *at least* 2, suppose w.l.o.g. $m \geq 2$. Then

$$|f(x_k, y_k)| = \left| \frac{x_k^n y_k^{m-2}}{1 + (x_k/y_k)^2} \right| \leq |x_k|^n |y_k|^{m-2}.$$

Now since $x_k \to 0$ then $x_k^n \to 0$, and since $m - 2 \geq 0$, then y_k^{m-2} either goes to zero or is identically 1. Thus $f(x_k, y_k) \to 0$ for all $m + n > 2$.
For the converse, suppose now that $m + n < 2$. We want to show that the limit does not exist, so we only have to find some sequence for which it does not converge. Let $x_k = y_k = 1/k$. Then

$$f(1/k, 1/k) = \frac{(\frac{1}{k})^{m+n}}{\frac{2}{k^2}} = \tfrac{1}{2} k^{2-m-n}$$

but, since $m + n < 2$, this is unbounded so the sequence $f(x_k, y_k)$ does not converge, and thus the limit does not exist.
Finally if $m + n = 2$, then the above sequence is identically $\frac{1}{2}$. But if we modify the sequence, say $x_k = 1/k$ and $y_k = 0$, then we get

$$f(1/k, 0) = 0$$

which is obviously zero. Since we get two different values on two sequences, the limit does not exist.

Problem 3: Let H be an infinite dimensional Hilbert space and $K \colon H \to H$ a compact linear operator. Prove the following statements.

(a) $0 \in \sigma(K)$, where $\sigma(K)$ is the spectrum of K.
(b) $\mathrm{Ker}(I - K) = \{0\}$ *iff* $\mathrm{Range}(I - K) = H$.
(c) $\sigma(K) = \sigma_p(K) \cup \{0\}$, where $\sigma_p(K)$ is the point spectrum of K.

Solution/Hint:

(a) We argue by contradiction, if $0 \notin \sigma(K)$ then K has a continuous inverse $K^{-1} : H \to H$ and thus $I = K \circ K^{-1}$. However, the composition of compact operators is compact, which implies that I has to be compact. This is a contradiction given that in infinite dimensional spaces the closed unit ball is not compact. Thus, $0 \in \sigma(K)$.
(b) Let $\mathrm{Ker}(I - K) = \{0\}$ then $I - K$ is one-to-one. Assume that $H_1 = \mathrm{Range}(I - K) \neq H$. Note that H_1 is a closed subspace of H because K is compact. Moreover, as $I - K$ is one-to-one then $H_2 = (I - K)H_1 \subset H_1$. Continue the construction for all n by defining

$$H_n = (I - K)^n H.$$

By induction we see that H_n is closed and $H \supset H_1 \supset H_2 \supset \dots$. Now, for each $n \geq 1$ choose a vector $e_n \in H_n \cap H_{n-1}^{\perp}$ with $\|e_n\| = 1$. Notice that if $m < n$ then

$$K(e_m - e_n) = -(e_m - Ke_m) + (e_n - Ke_n) + (e_m - e_n) = e_m + z_m$$

where $z_m = -(e_m - Ke_m) + (e_n - Ke_n) - e_n \in H_{m+1}$. Since $e_m \in H_{m+1}^{\perp}$, Pythagorias' Theorem implies that:

$$\|K(e_m - e_n)\| \geq \|e_m\| = 1$$

and the sequence $\{Ke_n\}_{n\geq 1}$ cannot have a strongly convergent subsequence, contradicting the compactness of K.

(c) Assume that $\lambda \in \sigma(K)$ with $\lambda \neq 0$. If $Ker(\lambda I - K) = \{0\}$, then by (b) $Range(\lambda I - K) = H$. Thus, by the Open Mapping Theorem $(\lambda I - K)$ has a bounded inverse, contradicting the assumption. We thus conclude that $\lambda \in \sigma_p(K)$.

Problem 4: Let $f(t)$ be a complex-valued function defined for $t \geq 0$. Its **Laplace Transform** Lf is a function defined on $s > 0$ given by

$$g(s) = (Lf)(s) = \int_0^\infty f(t)e^{-st}\, dt.$$

(a) Prove that L is a bounded linear map from $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$;

(b) Prove that

$$\|L\| \leq \sqrt{\pi}.$$

Hint: multiply and divide the integrand by $t^{1/4}$ to aid in integration.

Solution/Hint: We provide an upper bound on the L^2-norm of g using Cauchy-Schwarz inequality:

$$\begin{aligned}|g(s)|^2 = \left(\int_0^\infty f(t)e^{-st}\, dt\right)^2 &= \left[\int_0^\infty (f(t)e^{-\frac{st}{2}}t^{1/4})(e^{-\frac{st}{2}}t^{-1/4})\, dt\right]^2 \\ &\leq \int_0^\infty |f(t)|^2 e^{-st}t^{1/2}\, dt \int_0^\infty e^{-st}t^{-1/2}\, dt.\end{aligned}$$

We can now perform a change of variables, $u = st$ and $du = sdt$ and so

$$\int_0^\infty e^{-st}t^{-1/2}\, dt = s^{-1/2}\int_0^\infty e^{-u}u^{-1/2}\, du = Cs^{-1/2}.$$

Now, to compute C we use another change of variables: let $u = x^2$ and $du = 2xdx$

$$\int_0^\infty e^{-u}u^{-1/2}\, du = \int_0^\infty \frac{e^{-x^2}}{x}2x\, dx = 2\int_0^\infty e^{-x^2} = \sqrt{\pi}.$$

In summary, we have that

$$|g(s)|^2 \leq \sqrt{\pi}s^{-1/2}\int_0^\infty |f(t)|^2 e^{-st}t^{1/2}\, dt$$

and integrating over s gives:

$$\|g\|_2^2 = \int_0^\infty |g(s)|^2\, ds \leq \sqrt{\pi}\int_0^\infty \int_0^\infty |f(t)|^2 e^{-st}t^{1/2}s^{-1/2}\, dt\, ds.$$

After interchanging the oder of integration we see that:

$$\int_0^\infty e^{-st}t^{1/2}s^{-1/2}\, ds = \int_0^\infty e^{-u}u^{-1/2}\, du = \sqrt{\pi}$$

and so we get that

$$\|g\|_2^2 \leq \pi\|f\|_2^2.$$

From this we conclude that $\|L\| \leq \sqrt{\pi}$. This proves (a) and (b).

Please turn over

Problem 5: Let $A : H \to H$ be a compact and symmetric operator defined on a Hilbert space H. Define the *Rayleigh quotient* $R_A(x)$ as follows:

$$R_A(x) = \frac{(Ax, x)}{\|x\|^2}.$$

Denote the positive eigenvalues of A, indexed in decreasing order, by λ_k with $k = 1, 2, \dots,$ with corresponding eigenvectors z_n. Recall that we can compute

$$\lambda_N = \max_{x \perp \{z_1, z_2, \dots, z_{N-1}\}} \frac{(Ax, x)}{\|x\|^2}. \tag{1}$$

and the maximum is achieved by z_N.

(a) Prove that

$$\lambda_N = \max_{S_N} \left(\min_{x \in S_N} R_A(x) \right)$$

where S_N is any N-dimensional vector subspace of H.

(b) Prove that

$$\lambda_N = \min_{S_{N-1}} \left(\max_{x \perp \{S_{N-1}\}} R_A(x) \right)$$

where S_{N-1} is as in (a).

Solution/Hint:

(a) Since S_N is N-dimensional there exists a non-zero element $y \in S_N$ such that

$$(y, z_k) = 0,$$

for all $k = 1, 2, \dots, N-1$. By (1) we see that $R_A(y) \leq \lambda_N$. Moreover, as $y \in S_N$ we see that

$$\min_{S_N} R_A(x) \leq \lambda_N,$$

which holds for any N. On the other hand, let $S_N = span\{z_1, z_2, \dots, z_N\}$. The minimum of R_A on S_N, which is reached by the eigenvector z_N, is λ_N, which proves the result.

(b) Given any subspace S_{N-1} of dimension $N-1$, then $span\{z_1, z_2, \dots, z_N\}$ contains a vector y perpendicular to S_{N-1}. Moreover, for any $y \in span\{z_1, z_2, \dots, z_N\}$ we have that $R_A(y) \geq \lambda_N$, which implies that for S_{N-1}

$$\max_{x \perp S_{N-1}} R_A(x) \geq \lambda_N.$$

On the other hand, if we let $S_{N-1} = span\{z_1, z_n, \dots, z_{N-1}\}$ then by (1) equality holds in the above inequality, which proves the result.