Applied Analysis Preliminary Exam (Hints/solutions)
1:00 PM – 4:00 PM, August 21, 2023

Instructions You have three hours to complete this exam. Work all five problems; there are no optional problems. Each problem is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem or homework problem on the syllabus or discussed in class or in the Hunter & Nachtergaele book, unless you are directly proving such a result (when in doubt, ask the proctor). If you cannot finish part of a question, you may wish to move on to the next part; problems are graded with partial credit. **Write your student number on your exam, not your name**.

Problem 1 (20 points)

(a) Prove that $f(\boldsymbol{x}) = \|\boldsymbol{x}\|$ is continuous when $\|\cdot\|$ is a norm.

(b) Two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are said to be *equivalent* when there are two positive constants c and C such that $c\|\boldsymbol{x}\|_a \leq \|\boldsymbol{x}\|_b \leq C\|\boldsymbol{x}\|_a$ for all $\boldsymbol{x}$. Prove that all norms on $\mathbb{R}^n$ are equivalent. What part of this proof does not work in infinite dimensions? *Hint: Use part (a).*

Solution:

(a) Let ϵ be given and let $\delta = \epsilon$. Assume $\|\boldsymbol{x} - \boldsymbol{y}\| < \delta$. Then by the reverse triangle inequality

$$\big|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|\big| \leq \|\boldsymbol{x} - \boldsymbol{y}\| < \delta = \epsilon.$$

(b) If $\boldsymbol{x} = \boldsymbol{0}$ then the inequality holds trivially for any pair of norms and any pair of constants, so assume that $\boldsymbol{x} \neq \boldsymbol{0}$. The statement of equivalence of the two norms is equivalent to the statement that

$$c \leq \|\boldsymbol{u}\|_b \leq C$$

for every vector $\boldsymbol{u} = \boldsymbol{x}/\|\boldsymbol{x}\|_a$ that is a unit vector with respect to the a-norm. From the Heine-Borel theorem we know that the surface of the unit sphere in $\mathbb{R}^n$ is compact because it is closed and bounded. We also know that the norm $\|\cdot\|_b$ is a continuous function. It therefore achieves a maximum and a minimum on a compact set (the generalization of the extreme value theorem), and we can set

$$c = \min_{\|\boldsymbol{u}\|_a = 1} \|\boldsymbol{u}\|_b, \; C = \max_{\|\boldsymbol{u}\|_a = 1} \|\boldsymbol{u}\|_b.$$

To show that these are positive requires only noting that $c = 0$ implies that there is some vector $\boldsymbol{v}$ such that $\|\boldsymbol{v}\|_b = 0$ and $\|\boldsymbol{v}\|_a = 1$. This is not possible because $\|\boldsymbol{v}\|_b = 0 \Leftrightarrow \boldsymbol{v} = \boldsymbol{0}$, but $\|\boldsymbol{0}\|_a = 0 \neq 1$.

The unit sphere is not compact in infinite dimensions. We can replace max and min by sup and inf, but $c = \inf_{\|\boldsymbol{u}\|_a=1} \|\boldsymbol{u}\|_b$ may be zero, and $C = \sup_{\|\boldsymbol{u}\|_a=1} \|\boldsymbol{u}\|_b$ may be infinite.

Comments: This turned out to be a hard question! You could try to prove this with the open mapping theorem, which says that if $T : X \to Y$ is a bijective bounded linear map between Banach spaces, then T^{-1} is bounded too. Here, we let $X = Y = \mathbb{R}^n$ be the same space but with the $\|\cdot\|_a$ and $\|\cdot\|_b$ norms, respectively, and T be the identity. Then if you show T is bounded (which it must be, since it's linear on finite dimensional spaces), then we have the constant $C < \infty$, and the open mapping theorem gives the constant $c > 0$. However, this proof isn't great since it relies on the fact that finite dimensional linear operators are bounded, and that is really kind of the same statement as this problem, so it's circular.

Other common issues: because this is true in a normed space and not just a Hilbert space, the proof shouldn't use any kind of orthogonality (e.g., note that Parseval/Bessel's inequalities only work for the induced Hilbers space norm, not any norm).

Problem 2 (20 points) Consider the subset $\mathcal{F} \subset C[0,1]$ consisting of functions of the form

$$f(x) = \log(a - x)$$

for some $a \in [3, b]$ for some $b \in (3, \infty)$. Prove that $\mathcal{F}$ is a compact subset of $C[0,1]$.

Solution: We need to prove that the set is closed, bounded, and equicontinuous (and thus invoke the Arzela-Ascoli theorem).

- To show that it is closed, we will show that any limit point of a sequence within the set is also in the set. Any sequence of functions $f_n \in \mathcal{F}$ can be parameterized via

  $$f_n(x) = \log(a_n - x).$$

  If f_n is convergent with respect to the sup norm, then $f_n(x)$ is also convergent for any $x \in [0,1]$. Since e^x is a continuous function, we have that

  $$\exp\{f_\infty(x)\} = \exp\{\lim_{n\to\infty} f_n(x)\} = \lim_{n\to\infty} \exp\{f_n(x)\} = \lim_{n\to\infty} (a_n - x),$$

  which implies that $\lim_{n\to\infty} a_n$ exists and is equal to $x+ \exp\{f_\infty(x)\}$. Since $a_n \in [3, b]$ and $\{a_n\}$ is a convergent sequence, we infer that the limit a_∞ is also in $[3, b]$ since the interval $[3, b]$ is closed. Thus, for any $x \in [0,1]$

  $$\lim_{n\to\infty} f_n(x) = \log(a_\infty - x).$$

  Since $a_\infty \in [3, b]$ we have that every convergent sequence in $\mathcal{F}$ converges to a function that is also in $\mathcal{F}$, implying that $\mathcal{F}$ is closed.

  An alternative proof: use the fact that $(a_n) \subset [3, b]$ so it must have a convergent subsequence, and use the sequential continuity of log. (This is very similar to Exercise 1.27: if (x_n) is a sequence in a compact metric space and every convergent subsequence has the same limit x, then $x_n \to x$).

- To show boundedness: $\|f\|_\infty = \max_{x\in[0,1]} |\log(a - x)| = \max_{x\in[0,1]} \log(a - x) = \log(a) \le \log(b)$.

- To show equicontinuity we will show that all $f \in \mathcal{F}$ share the same Lipschitz constant. The derivative of an f in $\mathcal{F}$ is

  $$f'(x) = \frac{1}{x - a}.$$

  For $x \in [0,1]$ and $a \in [3, b]$ the absolute value is bounded above

  $$|f'(x)| \le \frac{1}{2}.$$

  All functions in $\mathcal{F}$ are Lipschitz continuous with constant $1/2$, so they are equicontinuous.

Problem 3 (20 points) Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint compact linear operator on a Hilbert space $\mathcal{H}$. Prove that $\text{ran}(A)$ is closed if and only if A is finite rank.

Solution: We'll start with the easy direction: let A be finite rank. Then $\text{ran}(A)$ is a subspace since A is linear, and is finite dimensional by assumption, hence must be closed (cf. Corollary 5.34 in Hunter & Nachtergaele). *Note: we didn't need compactness for this direction.*

Now the other direction: we'll suppose that A is not finite rank, and show $\text{ran}(A)$ cannot be closed. Using the implication of Thm. 8.17 in Hunter & Nachtergaele, we have the general result that

$$\mathcal{H} = \overline{\text{ran}(A)} \oplus \ker(A^*)$$

and since A is self-adjoint, in our case we have $\mathcal{H} = \overline{\text{ran}(A)} \oplus \ker(A)$. Define $\mathcal{H}_0 = \overline{\text{ran}(A)}$ which is a closed subspace of a Hilbert space and hence a Hilbert space itself, and define $A_0 : \mathcal{H}_0 \to \mathcal{H}$ the restriction of A to $\mathcal{H}_0$, so that A_0 is one-to-one. Clearly $\text{ran}(A) = \text{ran}(A_0)$ and it also follows that A_0 is compact.

For the sake of contradiction, let us assume $\text{ran}(A)$ is closed. Then the open mapping theorem (or Prop. 5.30 in Hunter & Nachtergaele) implies that $A_0^{-1} : \text{ran}(A) \to \mathcal{H}_0$ is bounded.

However, this leads to a contradiction. By the spectral theorem,

$$A_0 = \sum_{n=1}^{\infty} \lambda_n P_n$$

where $\lambda_n \to 0$ but infinitely many $\lambda_n \neq 0$ (else it would be finite-rank). Then consider a sequence $e_n \in \text{ran}(P_n)$ of unit-norm, and $A_0^{-1} e_n = \lambda_n^{-1} e_n$ which implies A_0^{-1} is unbounded.

Or, instead of the spectral theorem, you can do this. Let B be the unit ball in $\mathcal{H}$ which is compact iff $\mathcal{H}$ is finite dimensional. Since we're assuming A isn't finite rank, we've presupposed $\mathcal{H}$ is infinite dimensional. Now we'll prove that in fact B is compact, getting our contradiction. Take any sequence $(x_n) \subset B$, then since this is bounded and by compactness of A_0, we have that $(A_0 x_n)$ must be pre-compact, i.e., there is a convergent subsequence $(A_0 x_{n_k})$. But we showed that A_0^{-1} is bounded, hence (sequentially) continuous, so $(A_0^{-1} x_{n_k}) = (x_{n_k})$ has a convergent subsequence, hence B is (sequentially) compact, which is the contradiction.

Yet another variation. Consider $A_0 A^{-1} : \text{ran}(A) \to \text{ran}(A)$, which is a compact operator (A_0 being compact and A^{-1} being bounded by the open mapping theorem which applies if we're careful about the restriction to avoid the kernel of A, and since $\text{ran}(A)$ is closed). Yet $A_0 A^{-1}$ is the identity, and this is a compact operator iff the space is finite dimensional.

Problem 4 (20 points) Let E_k be measurable sets and define

$$\limsup E_k = \bigcap_{j=1}^{\infty} \left(\bigcup_{k=j}^{\infty} E_k \right), \quad \liminf E_k = \bigcup_{j=1}^{\infty} \left(\bigcap_{k=j}^{\infty} E_k \right).$$

(a) Fatou's lemma for functions says that if (f_n) is a sequence of non-negative measurable functions, then $\int \liminf f_n \leq \liminf \int f_n$. Prove the following variant of Fatou's lemma for measurable *sets*: $\mu(\liminf E_k) \leq \liminf \mu(E_k)$.

(b) Prove the *First Borel-Cantelli lemma*: $\sum \mu(E_k) < \infty$ implies $\mu(\limsup E_k) = 0$.

Solution: These all follow from the continuity of measure, namely, that if (A_i) is an increasing sequence of measurable sets, meaning that $A_i \subset A_{i+1}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i \right) = \lim_{i \to \infty} \mu(A_i).$$

This is exercise 12.3 in the book, and it is OK to state it without proof, but you can also prove it directly: due to the nesting, and defining $A_0 = \emptyset$ for notational convenience, we calculate:

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{\infty} A_i \right) = \mu\left(\bigcup_{i=1}^{\infty} (A_i \setminus A_{i-1}) \right) &= \sum_{i=1}^{\infty} \mu\left(A_i \setminus A_{i-1}\right) \quad \text{by countable additivity, since disjoint} \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} \mu\left(A_i \setminus A_{i-1}\right) \\
&= \lim_{n \to \infty} \mu\left(\bigcup_{i=1}^{n} A_i \setminus A_{i-1} \right) \\
&= \lim_{n \to \infty} \mu\left(A_n\right).
\end{aligned}$$

We also have this immediate corollary: If (A_i) is a decreasing sequence of measurable sets, meaning that $A_i \supset A_{i+1}$, and $\mu(A_k) < \infty$ for some k, then

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i).$$

A proof of this fact: wlog let A_1 have finite measure. Recall De Morgan's laws: $\cap A_i = (\cup_i A_i^c)^c$ where ^{c} represents complement with respect to a superset (such as A_1). Then

$$\mu\left(\cap A_i\right) = \mu(A_1) - \mu\left(\cup_i A_1 \setminus A_i\right)$$

and the sequence of sets $(A_1 \setminus A_i)$ is an increasing nested sequence, so we can apply the original continuity of measure result.

Now solutions in detail:

(a) We'll give two different styles of proof. The first proof uses the continuity of measure result used above. Let $F_j = \cap_{k=j}^{\infty} E_k$ so that $F_j \subset F_{j+1}$, i.e., the sets are nested. Also, fixing j, note that for all $k' \geq j$ then $\cap_{k\geq j} E_k \subset E_{k'}$ hence $\mu\left(\cap_{k\geq j} E_k\right) \leq \mu\left(E_{k'}\right)$, and if that is true for all $k' \geq j$ then it's also true for the infimum over all $k' \geq j$. Putting this altogether,

$$\begin{aligned}
\liminf \mu(E_k) &:= \lim_{j\to\infty}\left(\inf_{k'\geq j} \mu\left(E_{k'}\right)\right) \\
&\geq \lim_{j\to\infty}\left(\mu\left(\cap_{k\geq j} E_k\right)\right) \\
&= \lim_{j\to\infty}\left(\mu(F_j)\right) \\
&= \mu(\cup_j F_j) \\
&= \mu\left(\cup_j \cap_{k=j}^{\infty} E_k =: \liminf E_k\right).
\end{aligned}$$

An alternative style of proof is to deduce this from Fatou's lemma for functions. Let $\mathcal{I}_A$ denote the indicator function of a measurable set A, i.e.,

$$\mathcal{I}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

Then let $f_k = \mathcal{I}_{E_k}$ and it follows that $\int f_k d\mu = \mu(E_k)$. Then with a little bit of work, you can show that

$$\liminf f_k = \mathcal{I}_{\liminf E_k}$$

and from there, Fatou's lemma for functions gives the result.

(b) We'll use the corollary of the continuity of measure (i.e., the result for decreasing sets) and define $F_j = \cup_{k\geq j} E_k$ which is clearly decreasing, $F_{j+1} \subset F_j$. Furthermore,

$$\mu(F_1) = \mu\left(\cup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu\left(E_k\right)$$

which is finite by assumption, where we used countable sub-additivity (for non-disjoint sets) which follows pretty directly from the axioms for a measure (namely, non-negativity and countable additivity for disjoint sets).

Hence we're in position to apply the continuity of measure corollary. Calculate:

$$\begin{aligned}\mu\left(\limsup E_k\right) &= \mu\left(\lim_j F_j\right)\\ &= \lim_j \mu(F_j) \quad \text{by continuity of measure corollary}\\ &= \lim_j \mu\left(\cup_{k\geq j} E_k\right)\\ &\leq \lim_j \sum_{k\geq j} \mu\left(E_k\right) \text{by countable sub-additivity again}\\ &\leq 0\end{aligned}$$

as desired, where the last line follows because, again using that $\mu\left(E_k\right) < \infty$, it follows the limit of the tails of the sum must converge to zero (i.e., the partial sums must be Cauchy).

Problem 5 (20 points)

(a) Solve the following integro-differential equation by giving an expression for φ in terms of $f \in L^1(\mathbb{T})$

$$-\varphi''(x) + \frac{1}{\pi}\int_{\mathbb{T}} \cos^2\left(\frac{x-y}{2}\right)\varphi(y)\mathrm{d}y = f(x).$$

(b) Prove that if $f \in L^2(\mathbb{T})$ then $\varphi \in C^1(\mathbb{T})$.

Solution:

(a) Recognize

$$\frac{1}{\pi}\int_{\mathbb{T}} \cos^2\left(\frac{x-y}{2}\right)\varphi(y)\mathrm{d}y = (h * \varphi)(x)$$

where

$$h(x) = \frac{1}{\pi}\cos^2\left(\frac{x}{2}\right).$$

Taking the L^2 inner product of the integro-differential equation with $e^{inx}/\sqrt{2\pi}$ (i.e. integrate against $e^{-inx}/\sqrt{2\pi}$) yields

$$n^2\hat{\varphi}_n + \sqrt{2\pi}\hat{h}_n\hat{\varphi}_n = \hat{f}_n.$$

Solving for $\hat{\varphi}_n$ yields

$$\hat{\varphi}_n = \frac{\hat{f}_n}{n^2 + \sqrt{2\pi}\hat{h}_n}.$$

Our final expression for φ is

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\sum_n e^{inx}\frac{\hat{f}_n}{n^2 + \sqrt{2\pi}\hat{h}_n}.$$

In order for this to be well-defined, we need to make sure that $\sqrt{2\pi}\hat{h}_n \neq -n^2$ for any n. The Fourier coefficients of h are found from the power reduction formula

$$\cos^2\left(\frac{x}{2}\right) = \frac{1-\cos(x)}{2}.$$

So $\hat{h}_0 = \sqrt{\pi/2}$, $\hat{h}_{\pm 1} = \sqrt{\pi/8}$ and $\hat{h}_n = 0$ for $n \geq 2$.

(b) If $f \in L^2$ then $\varphi \in H^2$ because

$$\sum_n n^4|\hat{\varphi}_n|^2 = \sum_n \frac{n^4|\hat{f}_n|^2}{(n^2 + \sqrt{2\pi}\hat{h}_n)^2} \leq \sum_n |\hat{f}_n|^2 = \|f\|_2^2$$

using $\hat{h}_n \geq 0$.

By the Sobolev embedding theorem, $\varphi \in C^\ell(\mathbb{T})$ where ℓ is the largest integer strictly less than $2 - 1/2$, i.e. $\varphi \in C^1(\mathbb{T})$.