Applied Analysis Preliminary Exam

1:00 PM – 4:00 PM, August 21, 2023

Instructions You have three hours to complete this exam. Work all five problems; there are no optional problems. Each problem is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem or homework problem on the syllabus or discussed in class or in the Hunter & Nachtergaele book, unless you are directly proving such a result (when in doubt, ask the proctor). If you cannot finish part of a question, you may wish to move on to the next part; problems are graded with partial credit. **Write your student number on your exam, not your name**.

Problem 1 (20 points)

(a) Prove that $f(\boldsymbol{x}) = \|\boldsymbol{x}\|$ is continuous when $\|\cdot\|$ is a norm.

(b) Two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are said to be *equivalent* when there are two positive constants c and C such that $c\|\boldsymbol{x}\|_a \leq \|\boldsymbol{x}\|_b \leq C\|\boldsymbol{x}\|_a$ for all $\boldsymbol{x}$. Prove that all norms on $\mathbb{R}^n$ are equivalent. What part of this proof does not work in infinite dimensions? *Hint: Use part (a).*

Problem 2 (20 points) Consider the subset $\mathcal{F} \subset C[0,1]$ consisting of functions of the form

$$f(x) = \log(a - x)$$

for some $a \in [3, b]$ for some $b \in (3, \infty)$. Prove that $\mathcal{F}$ is a compact subset of $C[0,1]$.

Problem 3 (20 points) Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint compact linear operator on a Hilbert space $\mathcal{H}$. Prove that $\operatorname{ran}(A)$ is closed if and only if A is finite rank.

Problem 4 (20 points) Let E_k be measurable sets and define

$$\limsup E_k = \bigcap_{j=1}^{\infty} \left(\bigcup_{k=j}^{\infty} E_k \right), \quad \liminf E_k = \bigcup_{j=1}^{\infty} \left(\bigcap_{k=j}^{\infty} E_k \right).$$

(a) Fatou's lemma for functions says that if (f_n) is a sequence of non-negative measurable functions, then $\int \liminf f_n \leq \liminf \int f_n$. Prove the following variant of Fatou's lemma for measurable *sets*: $\mu(\liminf E_k) \leq \liminf \mu(E_k)$.

(b) Prove the *First Borel-Cantelli lemma*: $\sum \mu(E_k) < \infty$ implies $\mu(\limsup E_k) = 0$.

Problem 5 (20 points)

(a) Solve the following integro-differential equation by giving an expression for φ in terms of $f \in L^1(\mathbb{T})$

$$-\varphi''(x) + \frac{1}{\pi} \int_{\mathbb{T}} \cos^2\left(\frac{x-y}{2}\right) \varphi(y) \mathrm{d}y = f(x).$$

(b) Prove that if $f \in L^2(\mathbb{T})$ then $\varphi \in C^1(\mathbb{T})$.