Applied Analysis Preliminary Exam (Hints/solutions)
9:00 AM – 12:00 PM, August 17, 2021

Instructions You have three hours to complete this exam. Work all five problems; there are no optional problems. Each problem is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are directly proving such a theorem (when in doubt, ask the proctor). If you cannot finish part of a question, you may wish to move on to the next part; problems are graded with partial credit. **Write your student number on your exam, not your name**.

Problem 1 (20 points) The following two problems are unrelated.

(a) Let $\mathbf{r} \in \mathbb{R}^3$ and $\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{r^3}\mathbf{r}$ be a vector field where $r = \|\mathbf{r}\|_2$, and q and ϵ_0 are constants; you may recognize this as the electric field due to a charge q at the origin. By using the formula for the surface area of a sphere, observe that

$$\oiint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,dS = \frac{q}{\epsilon_0}$$

if S is a sphere of radius $R > 0$ (no work necessary on your part). Show that in fact this equation holds for *any* bounded, smooth closed surface S that encloses the origin (this is a simple version of Gauss' law). *Hint: is* $\mathbf{E}$ *divergence-free in some regions?*

Solution: Let S be our closed surface, and S_R be the sphere of radius R, and let V be the volume enclosed between S and S_R. To be very careful, let's ensure that S and S_R do not intersect; since S is bounded, simply choose R sufficiently large so that it completely encloses S.

Also note that $(\nabla\cdot\mathbf{E})(\mathbf{r}) = 0$ for all $\mathbf{r} \neq 0$ since divergence in spherical coordinates can be written

$$\nabla\cdot\mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi E_\phi) + \frac{1}{r\sin\phi}\frac{\partial E_\theta}{\partial\theta}$$

where (E_r, E_ϕ, E_θ) represent $\mathbf{E}$ in spherical coordinates. In our case, $E_r \propto r^{-2}$ and $E_\phi = E_\theta = 0$, hence $\nabla\cdot\mathbf{E} = 0$. You can also calculate the divergence directly in Cartesian coordinates if you wish: $\nabla\cdot\mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$ with $E_x = x/r^3$ and $\frac{\partial E_x}{\partial x} = \frac{r^3 - 3x^2 r}{r^6}$ when $r \neq 0$, and similarly for E_y and E_z, so when the partials are added together, they exactly cancel and give $\nabla\cdot\mathbf{E} = 0$.

Since V does *not* contain the origin,

$$\iiint_V \nabla\cdot\mathbf{E}\,dV = \iiint_V 0 = 0$$

and on the other hand, by the **divergence theorem**,

$$\begin{aligned}\iiint_V \nabla\cdot\mathbf{E}\,dV &= \oiint_{\partial V} \mathbf{E}\cdot\hat{\mathbf{n}}\,dS \\ &= \oiint_{S_R} \mathbf{E}\cdot\hat{\mathbf{n}}\,dS - \oiint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,dS \\ &= \frac{q}{\epsilon_0} - \oiint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,dS\end{aligned}$$

hence we conclude $\oiint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,dS = \frac{q}{\epsilon_0}$.

(b)

i. For every $n \in \mathbb{Z}$, show there is a unique solution $\alpha \in (-\pi/2, \pi/2]$ solving the equation $\cos(\alpha) = n^{\frac{3}{2}}\alpha$. *In the interest of time, you may give details for* $n = 1$ *and just a quick proof sketch for all other* n.

Solution: For $n = 0$, the unique solution is $\alpha = \pi/2$ by inspection. Note that $n^{\frac{3}{2}}$ does not have a well-defined real value if $n < 0$, so the question should have read $\cos(\alpha) = |n|^{\frac{3}{2}}\alpha$.

For $n = \pm 1$, we are solving the fixed point equation $F(\alpha) = \alpha$ for $F(\alpha) \stackrel{\text{def}}{=} \pm 1 \cos(\alpha)$. We'll show $n = +1$ in detail but $n = -1$ is very similar. Note: for $n = 1$, this is known as the Dottie number, and is a transcendental number near 0.739085.

Note $F(\alpha) \geq 0$ for all $\alpha \in (-\pi/2, \pi/2]$, hence we can exclude $\alpha < 0$ as a solution; similarly, we can exclude $\alpha > 1$, and thus focus on $\alpha \in [0, 1]$. Note $F(0) = 1$ and $F(1) \approx 0.54$ and F' is monotonically decreasing in $[0, \pi/2]$, so F maps $[0, 1]$ into $[0, 0.54]$, and F is continuous, so therefore we can apply the **contraction mapping theorem (aka Banach fixed point theorem)** if we can show F is a contraction. This is straightforward since $|F'(\alpha)| = |\sin(\alpha)| \leq \sin(1) < 1$ for $\alpha \in [0, 1]$ (note that just showing $|F'(\alpha)| < 1$ is not quite the same thing, as we need a uniform bound away from 1 to be a contraction). Note: there are other numbers you can use for this, e.g. observe $\cos([0, 1]) \subset \cos([0, \pi/3]) = [0, \frac{1}{2}]$ and $\sin([0, \frac{1}{2}]) \subset \sin([0, \pi/6]) = [0, \frac{1}{2}]$).

For $|n| \geq 2$, the argument is similar, applied to the fixed-point equation $F(\alpha) = \alpha$ with $F(\alpha) \stackrel{\text{def}}{=} \frac{1}{n^{\frac{3}{2}}} \cos(\alpha)$, which is continuous, and maps $[0, 1]$ into $[0, 1]$ (or $[-1, 0]$ into $[-1, 0]$ if $n < 0$), and has $\frac{-1}{n^{\frac{3}{2}}} \sin(\alpha)$ as its derivative so is clearly a contraction.

Comment: using the **intermediate value theorem** for continuous functions only guarantees existence, not uniqueness, so that was not a valid solution to this question, unless you added an argument about *strict monotonicity* of the root-finding problem.

ii. Let α_n be the unique solution in $(-\pi/2, \pi/2]$ to the equation $\cos(\alpha) = n^{\frac{3}{2}}\alpha$. Define the function

$$f(x) = \sum_{n \in \mathbb{Z}} \alpha_n \cos(nx).$$

Prove f is a continuous function. *Hint: what do you know about growth/decay of α_n?*

Solution: The equation is the Fourier series representation of f, with Fourier coefficients $|\hat{f}_n| \lesssim n^{-\frac{3}{2}}$ (they are not exactly $n^{-\frac{3}{2}}$ since this is the cosine series rather than with e^{inx}, and since we don't have an explicit expression for α_n). All we need from α_n is that it is bounded by $n^{-\frac{3}{2}}$; the fact that we don't have an explicit expression for it means that you likely cannot find a closed-form expression for f and use that to determine continuity.

Then observe that $\sum_n |n|^{2k} |\hat{f}_n|^2 \lesssim \sum_n |n|^{2k-3}$ is finite if $2k - 3 < -1$, i.e., if $k < 1$. Thus f is in the k^{th} Sobolev space on the torus, $H^k(\mathbb{T})$, for $k < 1$. In particular, this is true for $k = \frac{3}{4}$. Thus by the **Sobolev embedding theorem**, since $k > \frac{1}{2}$, it follows f **is continuous**. Careful: it is not necessarily true that $f \in H^1(\mathbb{T})$.

Comment: most attempts to prove continuity directly don't work. For example, if you derive $|f(x) - f(y)| \leq \sum_{n \in \mathbb{Z}} \alpha_n \cdot n \cdot |x - y|$ (which follows from sequential continuity of the absolute value function, the triangle inequality, and the fact that $\cos(nx)$ is Lipschitz continuous with constant n), the fact that $\alpha_n \lesssim n^{-\frac{3}{2}}$ isn't enough to make this a convergent series, since $\sum n^{-\frac{1}{2}} \geq \sum n^{-1} = \infty$. You can look at the partial sums and show uniform convergence (which would imply f is continuous, since the uniform limit of continuous functions is continuous), but if you do this, you're essentially reproving the Sobolev embedding theorem (see Lemma 7.8 in Hungter & Nachtergaele).

Problem 2 (20 points) Let $F = C([0, 1])$ be the Banach space of continuous real-valued functions on $[0, 1]$ equipped with the sup-norm, $\|\cdot\|_\infty$, and let $G = C^1([0, 1])$ be the Banach space of real-valued continuously differentiable functions on $[0, 1]$ equipped with the norm

$$\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty. \tag{1}$$

(a) Prove that $\|f\|_{C^1}$ defined by (1) is a norm on G.

(b) We note that $G \subset F$, and we consider the canonical injection

$$\mathcal{I} : (G, \|\cdot\|_{C^1}) \longrightarrow (F, \|\cdot\|_\infty) \tag{2}$$
$$f \longmapsto f \tag{3}$$

Prove that $\mathcal{I}$ is a compact operator. *Hint: you might consider using the Arzelà-Ascoli theorem.*

Solution:

(a) We have to check that $\|\cdot\|_{C^1}$ satisfies the three axioms that define a norm.

- If $f = 0$ then clearly $\|f\|_\infty = 0$, whence it follows that $f = 0$.
- We now prove the triangular inequality.

$$\begin{aligned} \|f+g\|_{C^1} &= \|f+g\|_\infty + \|f'+g'\|_\infty && (4) \\ &\le \|f\|_\infty + \|g\|_\infty + \|f'\|_\infty + \|g'\|_\infty && (5) \\ &\le \|f\|_{C^1} + \|g\|_{C^1}. && (6) \end{aligned}$$

- The third axiom is obvious.

Hence $\|\cdot\|$ is a norm.

(b) We wish to prove that $\mathcal{I}$ is a compact operator. Let K be the unit ball of G (for the norm $\|\cdot\|_{C^1}$), we need to prove that $\mathcal{I}(K)$ is precompact (relatively compact) in F. To do this we use Ascoli-Arzelà theorem. The subset $\mathcal{I}(K)$ is precompact in F if and only if it is bounded and equicontinuous.

- $\mathcal{I}(K)$ is bounded. Clearly, if $f \in K$, we have $\|f\|_{C^1} \le 1$ by definition of K, and thus

$$\|\mathcal{I}(f)\|_\infty = \|f\|_\infty \le \|f\|_{c^1} \le 1. \tag{7}$$

  We conclude that $\mathcal{I}(K)$ is bounded in F.
- $\mathcal{I}(K)$ is equicontinuous. Let $f \in K$, by the same argument as above we have $\|f'\|_\infty \le 1$ by definition of $\|\cdot\|_{C^1}$. Therefore,

$$\forall x, y \in [0,1], \quad |f(x) - f(y)| \le \sup_{z \in [0,1]} |f'(z)||x-y| \le |x-y| \tag{8}$$

  and thus $\mathcal{I}(K)$ is equicontinuous.

We conclude that $\mathcal{I}(K)$ is precompact, and therefore $\mathcal{I}$ is a compact operator.

Problem 3 (20 points) Let $F = C([0,1])$ be the Banach space of continuous real-valued functions on $[0,1]$ equipped with the sup-norm, $\|\cdot\|_\infty$. Let $u \in F$, and define the function Tu on $[0,1]$ by

$$Tu(x) = \begin{cases} \dfrac{1}{x}\displaystyle\int_0^x u(t)dt & \text{if } x \in (0,1], \\ u(0) & \text{if } x = 0. \end{cases} \tag{9}$$

(a) Prove that $Tu \in F$, and that

$$\forall u \in F, \quad \|Tu\| \le \|u\|. \tag{10}$$

Conclude that T belongs to $\mathcal{B}(F)$.

(b) Prove that the point spectrum (eigenvalues) of T is equal to $(0,1]$. Determine the corresponding eigenfunctions.

(c) Prove that $\|T\|_{\mathcal{B}(F)} = 1$.

Solution:

(a) The continuity of Tu at all $x > 0$ is clear (it's the product of continuous functions). Now we focus on proving it is continuous at 0. We note that if $x > 0$

$$Tu(x) = \frac{1}{x}\int_0^x u(t)dt = \int_0^1 u(xt)dt, \tag{11}$$

whence it follows that

$$\lim_{x\to 0} Tu(x) = \int_0^1 u(0)dt = u(0). \tag{12}$$

for the following reason: noting $|u|$ is bounded by $g(x) = \|u\|_\infty < \infty$, define $u_x = u(xt)$, then for all $x \le 1$, $(\forall t \in [0,1])\ |u_x(t)| \le g(t)$ and g is integrable over $[0,1]$, hence we can interchange the limit and the integral using **Lebesgue's Dominated Convergence Theorem**.

Alternative 1: for any $x > 0$, using $u(0) = \frac{1}{x}\int_0^x u(0)\,dt$,

$$\left|\big(Tu\big)(x) - u(0)\right| = \left|\frac{1}{x}\int_0^x u(t) - u(0)dt\right| \le \frac{1}{x}\int_0^x |u(t) - u(0)|\,dt \le \sup_{t\le x}|u(t) - u(0)|,$$

which can be made arbitrarily small by choosing x arbitrarily small (due to the continuity of u).

Alternative 2: note that $\lim_{x\to 0}\frac{1}{x}\int_0^x u(t)\,dt$ is of the form $0/0$, i.e., $\lim_{x\to 0}\frac{f(x)}{g(x)}$ with $f(x) = \int_0^x u(t)\,dt$ and $g(x) = x$. As f and g are differentiable, and as the limit $\lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{u(x)}{1} = u(0)$ exists (via the Fundamental Theorem of Calculus), we can use **L'Hôpital's** rule to conclude $\lim_{x\to 0}(Tu)(x) = u(0)$.

To show boundedness, clearly $|Tu(0)| \le \|u\|_\infty$, and if $x \ne 0$,

$$|Tu(x)| \le \int_0^1 \|u\|_\infty dt = \|u\|_\infty, \tag{13}$$

and thus

$$\|Tu\|_\infty \le \|u\|_\infty. \tag{14}$$

We conclude that T is a bounded, and (since it's clearly linear) therefore continuous, operator from F to itself.

(b) Let λ be an eigenvalue. First, we can show that $\lambda \ne 0$ because if u is the corresponding eigenfunction and $T(u) \equiv 0$ then $u(0) = 0$ and $\int_0^x u(t)\,dt = 0$ for all $x \in [0,1]$, so defining $F(x) = \int_0^x u(t)\,dt$, we have $F(x) = 0$, hence $F \equiv 0$, hence $F' \equiv 0$, but by the **fundamental theorem of calculus**, which applies since u is continuous, we have $F' = u$ (as continuous derivatives are unique), hence $u \equiv 0$, meaning that by definition $\lambda = 0$ is not an eigenvalue. Alternatively, to prove $u \equiv 0$ if $\lambda = 0$, note $\int_y^x u(t)\,dt = 0$ for all x, y but if there is some x_0 with $u(x_0) \ne 0$ then via continuity we have u of a fixed sign and bounded away from 0 on some interval $[x, y]$ and this leads to a contradiction.

We consider u the corresponding eigenfunction,

$$\frac{1}{x}\int_0^x u(t)dt = \lambda u(x). \tag{15}$$

The left-hand side of the equality is continuously differentiable on $(0,1]$. So we conclude that $u \in C^1((0,1])$. Multiplying by x and taking the derivative on both sides (and doing the product rule) yields

$$u(x) = \lambda u(x) + \lambda x u'(x). \tag{16}$$

Integrating this differential equation yields

$$u(x) = Cx^{-\alpha}, \quad \text{with} \quad \alpha = \frac{\lambda - 1}{\lambda} \quad \left(\text{i.e., } \lambda = \frac{1}{1-\alpha}\right), \tag{17}$$

where $C \in \mathbb{R}$. The condition that $u \in C^1((0,1])$ implies $\alpha < 0$, which further constraints λ. We have

$$0 < \lambda \le 1. \tag{18}$$

Reciprocally, if $\lambda \in (0,1]$, then $Cx^{-1+1/\lambda}$ is an eigenfunction associated with the eigenvalue λ.

(c) From (10), we have $\|T\|_{\mathcal{L}(F)} \le 1$, and the bound is achieved for $u(t) = 1$ or for any eigenfunction with eigenvalue $\lambda = 1$.

Problem 4 (20 points) Let X and Y be normed linear spaces, and $T : X \to Y$ a bounded linear transformation. Define the *transpose* $T^\top : Y^* \to X^*$ as the map that sends any $\phi \in Y^*$ to the linear functional $T^\top\phi \stackrel{\text{def}}{=} \phi \circ T \in X^*$, i.e., $(T^\top\phi)(x) = \phi(Tx)\,\forall x \in X$. Note that $T^\top$ is also a bounded linear transformation.

Let X and Y be Banach spaces, and suppose T is injective and has closed range. Prove that $T^\top$ is surjective. *Hint: you may wish to use major theorems, including the Hahn-Banach theorem.*

Solution: Because X and Y are Banach, we know by the **open mapping theorem** that since T is injective and has closed range, that there exists a constant $c > 0$ such that $(\forall x \in X)\, c\|x\| \leq \|Tx\|$. (This is the exact statement of **Prop. 5.30** in Hunter and Nachtergaele, and follows quite easily from the open mapping theorem since $\mathrm{ran}(T)$ is also a Banach space, so $T^{-1} : \mathrm{ran}(T) \to X$ is bounded).

Now to show that $T^\top$ is surjective, let $\psi \in X^*$, and we want to find $\phi \in Y^*$ such that $\psi = T^\top \phi$, i.e., $\psi = \phi \circ T$. Our plan will be to define such a ϕ, at least on $\mathrm{ran}(T)$. So let $y \in \mathrm{ran}(T)$, with $y = Tx$. Then define $\phi(y) \stackrel{\text{def}}{=} \psi(x)$, which is well-defined since x is unique because T is injective. Then it's easy to see ϕ is linear since ψ and T are linear. Furthermore, ϕ is bounded, since $\|x\| \leq c^{-1}\|y\|$ so $|\phi(y)| \leq c^{-1}\|\psi\|\|y\|$ (this is where we used the open-mapping result).

We've defined ϕ on $\mathrm{ran}(T) \subset Y$, but we need ϕ defined on all of Y so that we can claim $\phi \in Y^*$. By using the **Hahn-Banach theorem**, we know some extension of ϕ defined on all of Y must exist; with slight abuse of notation, we'll also refer to this extension as ϕ. The actual values of ϕ on $Y \setminus \mathrm{ran}(T)$ are not important since they do not affect $T^\top \phi$; all that matters is that ϕ is still bounded and linear. Hence $\psi = T^\top \phi$ as desired, meaning $T^\top$ is surjective.

Problem 5 (20 points) Let E be the inner-product space of complex-valued continuous functions on $[0, 1]$, equipped with the inner product

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}dt, \tag{19}$$

and the corresponding norm, $\|\cdot\|_2$. Let E_0 be the subspace of E composed of functions f with zero-mean,

$$\int_0^1 f(t) = 0. \tag{20}$$

We consider the following inner-product spaces H and H_0 defined respectively by

$$H = \{f \in E; f(1) = 0\}, \quad \text{and} \quad H_0 = E_0 \cap H = \left\{ f \in E; f(1) = \int_0^1 f(t)dt = 0 \right\}. \tag{21}$$

(a) Prove that H_0 is a proper closed subspace of H.

(b) Let $f_1 \in E_0$ defined by

$$\forall t \in [0, 1], \quad f_1(t) = t - \frac{1}{2}. \tag{22}$$

i. Prove that E is equal to the vector space spanned by f_1 and H.

ii. Prove that E_0 is equal to the vector space spanned by f_1 and H_0.

iii. Prove that f_1 is an element of the closure, $\overline{H_0}$, of H_0 in E.
Hint: you might construct a sequence of functions $u_n \in H_0$*, such that* $\lim_{n\to\infty} \|f_1 - u_n\|_2 = 0$*. You need not provide an explicit formula for* (u_n) *as long as it is clear they have the desired property.*

(c) Let $H_0^\perp$ be the orthogonal complement of H_0 in H. Let $g \in H_0^\perp$, and let $u_n \in H_0$, such that $\lim_{n\to\infty} \|f_1 - u_n\|_2 = 0$. Prove that

$$\left| \int_0^1 f_1(t)\overline{g(t)}dt \right| \leq \|f_1 - u_n\|_2 \, \|g\|_2. \tag{23}$$

Conclude that $\langle f_1, g \rangle = 0$.

(d) Show $H_0^\perp = E_0^\perp$ (where E_0 is the orthogonal complement of E_0 in E).

(e) Show that $E_0^\perp$ contains only constant functions, and then show $H_0^\perp = \{0\}$.

(f) Reconcile the apparent contradiction created by combining (a) and (e). To wit,

$$H_0^\perp \oplus H_0 = H_0 \neq H. \tag{24}$$

Solution:

(a) H_0 is the kernel (null space) of the bounded linear operator A defined by

$$Af = \langle f, f_0 \rangle \quad \text{where} \quad \forall t \in [0,1],\ f_0(t) = 1. \tag{25}$$

Cauchy-Schwarz yields

$$|Af| = |\langle f, f_0\rangle| \le \|f_0\|_2 \|f\|_2 = \|f\|_2 \tag{26}$$

where we have used $\|f_0\|_2 = 1$. We conclude that A is bounded and therefore continuous; its null space is therefore a closed subset of H. It is proper, since $\|A\| = 1$.

Alternative proof: That H_0 is a subspace is trivial; that it is proper can be seen by example, such as using $g(t) = 1 - t$ which is in $H \setminus H_0$. To show it is closed, one option is to use sequential continuity. Let $(f_n) \subset H_0$ be a convergent sequence with limit $f \in H$, i.e., $\|f - f_n\|_2 \to 0$. Note: we assume $f \in H$ since we asked if H_0 is a closed subspace *of* H. So we must show $f \in E_0$. To show $f \in E_0$,

$$\left|\int f\right| = \left|\int f_n + \int (f - f_n)\right| = \left|\int (f - f_n)\right| \le \|f_0\|_2 \|f - f_n\|_2$$

where f_0 is defined as in the earlier solution; and as before, this follows from Cauchy-Schwarz. Hence we can make $\int f$ arbitrarily close to 0, so it must be 0, hence $f \in E_0$. You may be tempted to show $\int f = 0$ by using (f_n) and Fatou's lemma (namely, $\int f = \int \lim f_n \le \liminf \int f_n = 0$, and similarly $\int -f = \ldots \le 0$), but you'd need to show that $\|f - f_n\|_2 \to 0$ implies *pointwise* convergence; this is false in L^2 but may be true in E though it'd be a bit of work to show it.

(b) i. Let $f \in E$, consider the function h defined by

$$h = f - 2f(1)f_1. \tag{27}$$

Clearly $h \in H$, $h(1) = f(1) - 2f(1)f_1(1) = f(1) - f(1) = 0$. We conclude that

$$\forall f \in E, \exists h \in H, \quad f = 2f(1)f_1 + h. \tag{28}$$

To wit, E is the vector space formed by the span of f_1 and H.

ii. Now, the mean of f_1 over $[0,1]$ is zero, that is f_1 is in E_0. But (27) implies that in this case, h also has zero-mean and thus $h \in H \cap E_0 = H_0$. In summary, if $f \in E_0$, (28) implies that E_0 is the vector space formed by the span of f_1 and H_0.

iii. Consider the sequence u_n that matches f_1 on $[1/n,\ 1 - 1/n]$, and has a continuous linear extension on $[0, 1/n]$ and $[1 - 1/n, 1]$, given by

$$u_n(t) = \begin{cases} \left(1 - \frac{n}{2}\right) t & \text{if } t \in [0, 1/n], \\ f_1(t) & \text{if } t \in [1/n,\ 1 - 1/n], \\ \left(1 - \frac{n}{2}\right)(t - 1) & \text{if } t \in [1 - 1/n, 1]. \end{cases} \tag{29}$$

The convergence of u_n to f_1 is a consequence of the following simple computation

$$\|f_1 - u_n\|_2 = \frac{1}{\sqrt{6n}}. \tag{30}$$

Finally, we can check that by construction $u_n(1) = 0$ and u_n is zero-mean, so $u_n \in H_0$. We conclude that the limit of u_n, f_1, is in $\overline{H_0}$.

(c) Let $g \in H_0^\perp$. The idea is to hit $f_1 - u_n$ against g, and pass to the limit. We have

$$\left|\int_0^1 f_1(t)\overline{g(t)}dt\right| = \left|\int_0^1 (f_1(t) - u_n(t))\,\overline{g(t)}dt\right| \le \|f_1 - u_n\|_2\ \|g\|_2 \tag{31}$$

where we have used the fact that $\langle u_n, g\rangle = 0$ to get to the second term, and Cauchy-Schwarz yields the last inequality. Now because of (30), we conclude that

$$\forall n \in \mathbb{N}, \quad \left|\int_0^1 f_1(t)\overline{g(t)}dt\right| \le \frac{\|g\|_2}{\sqrt{6n}}. \tag{32}$$

in other words,

$$\int_0^1 f_1(t)\overline{g(t)}dt = 0. \tag{33}$$

(d) Because H_0 is a subspace of E_0, we have $E_0^\perp \subset H_0^\perp$. Now let $g \in H_0^\perp$, and let $g \in E_0$. Because E_0 is the span of f_1 and H_0, there exist $\alpha \in \mathbb{C}$ and $h_0 \in H_0$ such that,

$$g = \alpha f_1 + h_0. \tag{34}$$

Because $\langle g, h_0 \rangle = 0$ ($g \in H_0^\perp$) and $\langle g, f_1 \rangle = 0$ (see previous question), we conclude that

$$\langle g, g \rangle = \langle g, \alpha f_1 + h_0 \rangle = \overline{\alpha} \langle g, f_1 \rangle + \langle g, h_0 \rangle = 0. \tag{35}$$

We conclude that $g \in E_0^\perp$, and thus $E_0^\perp = H_0^\perp$.

(e) We know that $E_0 = \{f_0\}^\perp$, so $E_0^\perp = \overline{\text{span}\,\{f_0\}} = \text{span}\,\{f_0\}$, where all the orthogonal complements are taken in E. Therefore, $E_0^\perp$ contains only constant function.

Let $g \in H_0^\perp$, then $g \in H_0^\perp \subset H$, since $H_0^\perp$ is the orthogonal complement of H_0 in H, and thereforee $g(1) = 0$. Now, $g \in E_0^\perp = H_0^\perp$, and thus g is constant, so $g(t) = g(1) = 0$, and we conclude that $g = 0$. In summary, $H_0^\perp = \{0\}$.

i. Alternative proof that $g \in E_0^\perp$ is constant. For shorthand, let $\mu = \langle g, f_0 \rangle$ be the mean, and $g_0 = g - \mu f_0$ as above. Then

$$\|g_0\|^2 = \langle g - \mu f_0, g - \mu f_0 \rangle = \langle g, g - \mu f_0 \rangle - \langle \mu f_0, g - \mu f_0 \rangle = -\langle \mu f_0, g - \mu f_0 \rangle$$

since $\langle g, g - \mu f_0 \rangle = 0$ because $g - \mu f_0 \in E_0$ and $g \in E_0^\perp$. Thus

$$\|g_0\|^2 = -\langle \mu f_0, g - \mu f_0 \rangle = |\mu|^2 \|f_0\|^2 - \mu \langle f_0, g \rangle = |\mu|^2 - |\mu|^2 = 0$$

hence $g_0 = 0$, i.e., g is a constant function (it doesn't deviate from its mean).

ii. Second alternative proof that $g \in E_0^\perp$ is constant. We define g_0 to be the function g centered around its mean,

$$g_0 = g - \int_0^1 g(t)dt, = g - \langle g, f_0 \rangle f_0, \tag{36}$$

where we recall, that $f_0(t) = 1, \forall t \in [0,1]$. By construction, $g_0 \in E_0$. Since $g \in E_0^\perp$, we have

$$0 = \langle g, g_0 \rangle = \langle g, g - \langle g, f_0 \rangle f_0 \rangle = \|g\|_2^2 - |\langle g, f_0 \rangle|^2 = \|g\|_2^2 \, \|f_0\|_2^2 - |\langle g, f_0 \rangle|^2 . \tag{37}$$

In short,

$$\|g\|_2^2 \, \|f_0\|_2^2 = |\langle g, f_0 \rangle|^2 . \tag{38}$$

We recognize the form of Cauchy-Schwartz, when the two vectors g and f_0 are colinear. We conclude that g is constant.

(f) The apparent contradiction stems from the fact that we work within a pre-Hilbert space, H. But H is not complete, so it is not a Hilbert space. Despite the fact that $H_0^\perp = \{0\}$, we only have

$$H_0 = \overline{H_0} \neq H. \tag{39}$$

For instance $f(t) = 1 - t$ is in H, but not in H_0.