Applied Analysis Preliminary Exam (Hints/solutions)
9:00 AM – 12:00 PM, August 17, 2021

Instructions You have three hours to complete this exam. Work all five problems; there are no optional problems. Each problem is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are directly proving such a theorem (when in doubt, ask the proctor). If you cannot finish part of a question, you may wish to move on to the next part; problems are graded with partial credit. Write your student number on your exam, not your name.

Problem 1 (20 points) The following two problems are unrelated.

(a) Let \( r \in \mathbb{R}^3 \) and \( \mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r}}{r} \) be a vector field where \( r = ||\mathbf{r}||_2 \), and \( q \) and \( \epsilon_0 \) are constants; you may recognize this as the electric field due to a charge \( q \) at the origin. By using the formula for the surface area of a sphere, observe that
\[
\iiint_S \mathbf{E} \cdot \hat{n} dS = \frac{q}{\epsilon_0}
\]
if \( S \) is a sphere of radius \( R > 0 \) (no work necessary on your part). Show that in fact this equation holds for any bounded, smooth closed surface \( S \) that encloses the origin (this is a simple version of Gauss’ law). Hint: is \( \mathbf{E} \) divergence-free in some regions?

Solution: Let \( S \) be our closed surface, and \( S_R \) be the sphere of radius \( R \), and let \( V \) be the volume enclosed between \( S \) and \( S_R \). To be very careful, let’s ensure that \( S \) and \( S_R \) do not intersect; since \( S \) is bounded, simply choose \( R \) sufficiently large so that it completely encloses \( S \).

Also note that \( (\nabla \cdot \mathbf{E})(\mathbf{r}) = 0 \) for all \( \mathbf{r} \neq 0 \) since divergence in spherical coordinates can be written
\[
\nabla \cdot \mathbf{E} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r) + \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} (\sin \phi E_\phi) + \frac{1}{r \sin \phi} \frac{\partial E_\theta}{\partial \theta}
\]
where \( (E_r, E_\theta, E_\phi) \) represent \( \mathbf{E} \) in spherical coordinates. In our case, \( E_r \propto r^{-2} \) and \( E_\phi = E_\theta = 0 \), hence \( \nabla \cdot \mathbf{E} = 0 \). You can also calculate the divergence directly in Cartesian coordinates if you wish: \( \nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \) with \( E_x = x/r^3 \) and \( \frac{\partial E_x}{\partial x} = r^3 - 3x^2r^{-5} \) when \( r \neq 0 \), and similarly for \( E_y \) and \( E_z \), so when the partials are added together, they exactly cancel and give \( \nabla \cdot \mathbf{E} = 0 \).

Since \( V \) does not contain the origin,
\[
\iiint_V \nabla \cdot \mathbf{E} dV = \iiint_V 0 = 0
\]
and on the other hand, by the divergence theorem,
\[
\iiint_V \nabla \cdot \mathbf{E} dV = \int_{\partial V} \mathbf{E} \cdot \hat{n} dS
= \int_{S_R} \mathbf{E} \cdot \hat{n} dS - \int_{S} \mathbf{E} \cdot \hat{n} dS
= \frac{q}{\epsilon_0} - \iiint_S \mathbf{E} \cdot \hat{n} dS
\]

hence we conclude \( \int_{S} \mathbf{E} \cdot \hat{n} dS = \frac{q}{\epsilon_0} \).

(b)

i. For every \( n \in \mathbb{Z} \), show there is a unique solution \( \alpha \in (-\pi/2, \pi/2) \) solving the equation \( \cos(\alpha) = n^{3/2} \alpha \).

In the interest of time, you may give details for \( n = 1 \) and just a quick proof sketch for all other \( n \).

Solution: For \( n = 0 \), the unique solution is \( \alpha = \pi/2 \) by inspection. Note that \( n^{3/2} \) does not have a well-defined real value if \( n < 0 \), so the question should have read \( \cos(\alpha) = |n|^{3/2} \alpha \).
For $n = \pm 1$, we are solving the fixed point equation $F(\alpha) = \alpha$ for $F(\alpha) \overset{\text{def}}{=} \pm 1 \cos(\alpha)$. We’ll show $n = +1$ in detail but $n = -1$ is very similar. Note: for $n = 1$, this is known as the Dottie number, and is a transcendental number near 0.739085.

Note $F(\alpha) \geq 0$ for all $\alpha \in (-\pi/2, \pi/2)$, hence we can exclude $\alpha < 0$ as a solution; similarly, we can exclude $\alpha > 1$, and thus focus on $\alpha \in [0,1]$. Note $F(0) = 1$ and $F(1) \approx 0.54$ and $F'$ is monotonically decreasing in $[0, \pi/2]$, so $F$ maps $[0,1]$ into $[0,0.54]$, and $F$ is continuous, so therefore we can apply the contraction mapping theorem (aka Banach fixed point theorem) if we can show $F$ is a contraction. This is straightforward since $|F'(\alpha)| = |\sin(\alpha)| \leq \sin(1) < 1$ for $\alpha \in [0,1]$ (note that just showing $F'(\alpha) < 1$ is not quite the same thing, as we need a uniform bound away from 1 to be a contraction). Note: there are other numbers you can use for this, e.g. observe $\cos([0,1]) \subset \cos([0,\pi/3]) = [0, \frac{1}{2}]$ and $\sin([0,\frac{1}{2}]) \subset \sin([0,\pi/6]) = [0, \frac{1}{2}]$.

For $|n| \geq 2$, the argument is similar, applied to the fixed-point equation $F(\alpha) = \alpha$ with $F(\alpha) \overset{\text{def}}{=} \frac{1}{n^\frac{2}{3}} \sin(\alpha)$, which is continuous, and maps $[0,1]$ into $[0,1]$ (or $[-1,0]$ into $[-1,0]$ if $n < 0$, and has $\frac{1}{n^\frac{2}{3}} \sin(\alpha)$ as its derivative so is clearly a contraction.

Comment: using the intermediate value theorem for continuous functions only guarantees existence, not uniqueness, so that was not a valid solution to this question, unless you added an argument about strict monotonicity of the root-finding problem.

ii. Let $\alpha_n$ be the unique solution in $(-\pi/2, \pi/2)$ to the equation $\cos(\alpha) = n^\frac{2}{3} \alpha$. Define the function

$$f(x) = \sum_{n \in \mathbb{Z}} \alpha_n \cos(nx).$$

Prove $f$ is a continuous function. Hint: what do you know about growth/decay of $\alpha_n$?

Solution: The equation is the Fourier series representation of $f$, with Fourier coefficients $|\hat{f}_n| \lesssim n^{-\frac{3}{2}}$ (they are not exactly $n^{-\frac{3}{2}}$ since this is the cosine series rather than with $e^{inx}$, and since we don’t have an explicit expression for $\alpha_n$). All we need from $\alpha_n$ is that it is bounded by $n^{-\frac{3}{2}}$; the fact that we don’t have an explicit expression for it means that you likely cannot find a closed-form expression for $f$ and use that to determine continuity.

Then observe that $\sum_n |n|^{2k} |\hat{f}_n|^2 \lesssim \sum_n |n|^{2k-3}$ is finite if $2k - 3 < -1$, i.e., if $k < 1$. Thus $f$ is in the $k^{th}$ Sobolev space on the torus, $H^k(\mathbb{T})$, for $k < 1$. In particular, this is true for $k = \frac{3}{4}$. Thus by the Sobolev embedding theorem, since $k > \frac{1}{2}$, it follows $f$ is continuous. Careful: it is not necessarily true that $f \in H^1(\mathbb{T})$.

Comment: most attempts to prove continuity directly don’t work. For example, if you derive $|f(x) - f(y)| \leq \sum_{n \in \mathbb{Z}} \alpha_n \cdot |n| \cdot |x - y|$ (which follows from sequential continuity of the absolute value function, the triangle inequality, and the fact that $\cos(nx)$ is Lipschitz continuous with constant $n$), the fact that $\alpha_n \lesssim n^{-\frac{3}{2}}$ isn’t enough to make this a convergent series, since $\sum n^{-\frac{3}{2}} \geq \sum n^{-1} = \infty$. You can look at the partial sums and show uniform convergence (which would imply $f$ is continuous, since the uniform limit of continuous functions is continuous), but if you do this, you’re essentially reproving the Sobolev embedding theorem (see Lemma 7.8 in Hunger & Nachtergaele).

**Problem 2** **(20 points)** Let $F = C([0,1])$ be the Banach space of continuous real-valued functions on $[0,1]$ equipped with the sup-norm, $\| \cdot \|_\infty$, and let $G = C^1([0,1])$ be the Banach space of real-valued continuously differentiable functions on $[0,1]$ equipped with the norm

$$\| f \|_C^1 = \| f \|_\infty + \| f' \|_\infty.$$  \hspace{1cm} (1)

(a) Prove that $\| f \|_C^1$ defined by (1) is a norm on $G$.

(b) We note that $G \subset F$, and we consider the canonical injection

$$I : (G, \| \cdot \|_C^1) \longrightarrow (F, \| \cdot \|_\infty)$$

$$f \longmapsto f \hspace{1cm} (2)$$

Prove that $I$ is a compact operator. Hint: you might consider using the Arzelà-Ascoli theorem.

Solution:
(a) We have to check that \( \| \cdot \|_{C^1} \) satisfies the three axioms that define a norm.

- If \( f = 0 \) then clearly \( \| f \|_{\infty} = 0 \), whence it follows that \( f = 0 \).
- We now prove the triangular inequality.

\[
\| f + g \|_{C^1} = \| f + g \|_{\infty} + \| f' + g' \|_{\infty} \\
\leq \| f \|_{\infty} + \| g \|_{\infty} + \| f' \|_{\infty} + \| g' \|_{\infty} \\
\leq \| f \|_{C^1} + \| g \|_{C^1}.
\]

(b) We wish to prove that \( I \) is a compact operator. Let \( K \) be the unit ball of \( G \) (for the norm \( \| \cdot \|_{C^1} \)), we need to prove that \( I(K) \) is precompact (relatively compact) in \( F \). To do this we use Ascoli-Arzelà theorem. The subset \( I(K) \) is precompact in \( F \) if and only if it is bounded and equicontinuous.

- \( I(K) \) is bounded. Clearly, if \( f \in K \), we have \( \| f \|_{C^1} \leq 1 \) by definition of \( K \), and thus

\[
\| I(f) \|_{\infty} = \| f \|_{\infty} \leq \| f \|_{C^1} \leq 1.
\]

We conclude that \( I(K) \) is bounded in \( F \).

- \( I(K) \) is equicontinuous. Let \( f \in K \), by the same argument as above we have \( \| f' \|_{\infty} \leq 1 \) by definition of \( \| \cdot \|_{C^1} \). Therefore,

\[
\forall x,y \in [0,1], \quad |f(x) - f(y)| \leq \sup_{z \in [0,1]} |f'(z)||x-y| \leq |x-y|
\]

and thus \( I(K) \) is equicontinuous.

We conclude that \( I(K) \) is precompact, and therefore \( I \) is a compact operator.

**Problem 3 (20 points)** Let \( F = C([0,1]) \) be the Banach space of continuous real-valued functions on \([0,1]\) equipped with the sup-norm, \( \| \cdot \|_{\infty} \). Let \( u \in F \), and define the function \( Tu \) on \([0,1]\) by

\[
Tu(x) = \begin{cases} 
\frac{1}{x} \int_0^x u(t)dt & \text{if } x \in (0,1], \\
u(0) & \text{if } x = 0.
\end{cases}
\]

(a) Prove that \( Tu \in F \), and that

\[
\forall u \in F, \quad \| Tu \| \leq \| u \|.
\]

Conclude that \( T \) belongs to \( B(F) \).

(b) Prove that the point spectrum (eigenvalues) of \( T \) is equal to \((0,1]\). Determine the corresponding eigenfunctions.

(c) Prove that \( \| T \|_{B(F)} = 1 \).

**Solution:**

(a) The continuity of \( Tu \) at all \( x > 0 \) is clear (it’s the product of continuous functions). Now we focus on proving it is continuous at \( 0 \). We note that if \( x > 0 \)

\[
Tu(x) = \frac{1}{x} \int_0^x u(t)dt = \int_0^1 u(xt)dt,
\]

whence it follows that

\[
\lim_{x \to 0} Tu(x) = \int_0^1 u(0)dt = u(0).
\]

for the following reason: noting \( |u| \) is bounded by \( g(x) = \| u \|_{\infty} < \infty \), define \( u_x = u(xt) \), then for all \( x \leq 1 \), \( \forall t \in [0,1] \) \( u_x(t) \leq g(t) \) and \( g \) is integrable over \([0,1]\), hence we can interchange the limit and the integral using Lebesgue's Dominated Convergence Theorem.
Alternative 1: for any $x > 0$, using $u(0) = \frac{1}{x} \int_0^x u(0) \, dt$, 

$$|(Tu)(x) - u(0)| = \frac{1}{x} \int_0^x u(t) - u(0) \, dt \leq \frac{1}{x} \int_0^x |u(t) - u(0)| \, dt \leq \sup_{t \leq x} |u(t) - u(0)|,$$

which can be made arbitrarily small by choosing $x$ arbitrarily small (due to the continuity of $u$).

Alternative 2: note that $\lim_{x \to 0} \frac{1}{x} \int_0^x u(t) \, dt$ is of the form $0/0$, i.e., $\lim_{x \to 0} \frac{f(x)}{g(x)}$ with $f(x) = \int_0^x u(t) \, dt$ and $g(x) = x$. As $f$ and $g$ are differentiable, and as the limit $\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{u(x)}{1} = u(0)$ exists (via the Fundamental Theorem of Calculus), we can use L'Hôpital's rule to conclude $\lim_{x \to 0} (Tu)(x) = u(0)$.

To show boundedness, clearly $|Tu(0)| \leq \|u\|_{\infty}$, and if $x \neq 0$,

$$|Tu(x)| \leq \int_0^1 \|u\|_{\infty} \, dt = \|u\|_{\infty},$$

and thus

$$\|Tu\|_{\infty} \leq \|u\|_{\infty}. \quad (14)$$

We conclude that $T$ is a bounded, and (since it's clearly linear) therefore continuous, operator from $F$ to itself.

(b) Let $\lambda$ be an eigenvalue. First, we can show that $\lambda \neq 0$ because if $u$ is the corresponding eigenfunction and $T(u) \equiv 0$ then $u(0) = 0$ and $\int_0^x u(t) \, dt = 0$ for all $x \in [0, 1]$, so defining $F(x) = \int_0^x u(t) \, dt$, we have $F(x) = 0$, hence $F \equiv 0$, but by the fundamental theorem of calculus, which applies since $u$ is continuous, we have $F' = u$ (as continuous derivatives are unique), hence $u \equiv 0$, meaning that by definition $\lambda = 0$ is not an eigenvalue. Alternatively, to prove $u \equiv 0$ if $\lambda = 0$, note $\int_0^x u(t) \, dt = 0$ for all $x, y$ but if there is some $x_0$ with $u(x_0) \neq 0$ then via continuity we have $u$ of a fixed sign and bounded away from 0 on some interval $[x, y]$ and this leads to a contradiction.

We consider $u$ the corresponding eigenfunction,

$$\frac{1}{x} \int_0^x u(t) \, dt = \lambda u(x). \quad (15)$$

The left-hand side of the equality is continuously differentiable on $(0, 1]$. So we conclude that $u \in C^1((0, 1])$. Multiplying by $x$ and taking the derivative on both sides (and doing the product rule) yields

$$u(x) = \lambda u(x) + \lambda xu'(x).$$

Integrating this differential equation yields

$$u(x) = Cx^{-\alpha}, \quad \text{with} \quad \alpha = \frac{\lambda - 1}{\lambda} \quad \left(\text{i.e., } \lambda = \frac{1}{1 - \alpha}\right), \quad (17)$$

where $C \in \mathbb{R}$. The condition that $u \in C^1((0, 1])$ implies $\alpha < 0$, which further constraints $\lambda$. We have

$$0 < \lambda \leq 1. \quad (18)$$

Reciprocally, if $\lambda \in (0, 1]$, then $Cx^{-1+1/\lambda}$ is an eigenfunction associated with the eigenvalue $\lambda$.

(c) From (10), we have $\|T\|_{L(F)} \leq 1$, and the bound is achieved for $u(t) = 1$ or for any eigenfunction with eigenvalue $\lambda = 1$.

**Problem 4 (20 points)** Let $X$ and $Y$ be normed linear spaces, and $T : X \to Y$ a bounded linear transformation. Define the transpose $T^\top : Y^* \to X^*$ as the map that sends any $\phi \in Y^*$ to the linear functional $T^\top \phi \overset{\text{def}}{=} \phi \circ T \in X^*$, i.e., $(T^\top \phi)(x) = \phi(Tx) \forall x \in X$. Note that $T^\top$ is also a bounded linear transformation. Let $X$ and $Y$ be Banach spaces, and suppose $T$ is injective and has closed range. Prove that $T^\top$ is surjective. *Hint: you may wish to use major theorems, including the Hahn-Banach theorem.*
Solution: Because $X$ and $Y$ are Banach, we know by the open mapping theorem that since $T$ is injective and has closed range, that there exists a constant $c > 0$ such that $(\forall x \in X) \ c\|x\| \leq \|Tx\|$. (This is the exact statement of Prop. 5.30 in Hunter and Nachtergaele, and follows quite easily from the open mapping theorem since $\text{ran}(T)$ is also a Banach space, so $T^{-1}: \text{ran}(T) \to X$ is bounded).

Now to show that $T^T$ is surjective, let $\psi \in X^*$, and we want to find $\phi \in Y^*$ such that $\psi = T^T \phi$, i.e., $\psi = \phi \circ T$. Our plan will be to define such a $\phi$, at least on $\text{ran}(T)$. So let $y \in \text{ran}(T)$, with $y = Tx$. Then define $\phi(y) \overset{\text{def}}{=} \psi(x)$, which is well-defined since $x$ is unique because $T$ is injective. Then it’s easy to see $\phi$ is linear since $\psi$ and $T$ are linear. Furthermore, $\phi$ is bounded, since $\|x\| \leq c^{-1}\|y\|$ so $\|\phi(y)\| \leq c^{-1}\|\psi\|\|y\|$ (this is where we used the open-mapping result).

We’ve defined $\phi$ on $\text{ran}(T) \subset Y$, but we need $\phi$ defined on all of $Y$ so that we can claim $\phi \in Y^*$. By using the Hahn-Banach theorem, we know some extension of $\phi$ defined on all of $Y$ must exist; with slight abuse of notation, we’ll also refer to this extension as $\phi$. The actual values of $\phi$ on $Y\setminus \text{ran}(T)$ are not important since they do not affect $T^T \phi$; all that matters is that $\phi$ is still bounded and linear. Hence $\psi = T^T \phi$ as desired, meaning $T^T$ is surjective.

**Problem 5 (20 points)** Let $E$ be the inner-product space of complex-valued continuous functions on $[0, 1]$, equipped with the inner product

$$
\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} \, dt,
$$

and the corresponding norm, $\| \cdot \|_2$. Let $E_0$ be the subspace of $E$ composed of functions $f$ with zero-mean,

$$
\int_0^1 f(t) = 0.
$$

We consider the following inner-product spaces $H$ and $H_0$ defined respectively by

$$
H = \{ f \in E; f(1) = 0 \}, \quad \text{and} \quad H_0 = E_0 \cap H = \left\{ f \in E; f(1) = \int_0^1 f(t) \, dt = 0 \right\}.
$$

(a) Prove that $H_0$ is a proper closed subspace of $H$.
(b) Let $f_1 \in E_0$ defined by

$$
\forall t \in [0, 1], \quad f_1(t) = t - \frac{1}{2}.
$$

i. Prove that $E$ is equal to the vector space spanned by $f_1$ and $H$.
ii. Prove that $E_0$ is equal to the vector space spanned by $f_1$ and $H_0$.
iii. Prove that $f_1$ is an element of the closure, $\overline{H_0}$, of $H_0$ in $E$.

*Hint: you might construct a sequence of functions $u_n \in H_0$, such that $\lim_{n \to \infty} \|f_1 - u_n\|_2 = 0$. You need not provide an explicit formula for $(u_n)$ as long as it is clear they have the desired property.*

(c) Let $H_0^\perp$ be the orthogonal complement of $H_0$ in $H$. Let $g \in H_0^\perp$, and let $u_n \in H_0$, such that

$$
\lim_{n \to \infty} \|f_1 - u_n\|_2 = 0.
$$

Prove that

$$
\int_0^1 |f_1(t) \overline{g(t)}| \, dt \leq \|f_1 - u_n\|_2 \|g\|_2.
$$

Conclude that $\langle f_1, g \rangle = 0$.
(d) Show $H_0^\perp = E_0^\perp$ (where $E_0$ is the orthogonal complement of $E_0$ in $E$).
(e) Show that $E_0^\perp$ contains only constant functions, and then show $H_0^\perp = \{0\}$.
(f) Reconcile the apparent contradiction created by combining (a) and (e). To wit,

$$
H_0^\perp \oplus H_0 = H_0 \neq H.
$$

**Solution:**
(a) $H_0$ is the kernel (null space) of the bounded linear operator $A$ defined by
\[ Af = \langle f, f_0 \rangle \quad \text{where} \quad \forall t \in [0, 1], \; f_0(t) = 1. \] (25)

**Cauchy-Schwarz** yields
\[ |Af| = |\langle f, f_0 \rangle| \leq \|f_0\|_2 \|f\|_2 = \|f\|_2 \] (26)
where we have used $\|f_0\|_2 = 1$. We conclude that $A$ is bounded and therefore continuous; its null space is therefore a closed subset of $H$. It is proper, since $\|A\| = 1$.

**Alternative proof:** That $H_0$ is a subspace is trivial; that it is proper can be seen by example, such as using $g(t) = 1 - t$ which is in $H \setminus H_0$. To show it is closed, one option is to use sequential continuity. Let $(f_n) \subset H_0$ be a convergent sequence with limit $f \in H$, i.e., $\|f - f_n\|_2 \to 0$. Note: we assume $f \in H$ since we asked if $H_0$ is a closed subspace of $H$. So we must show $f \in E_0$. To show $f \in E_0$,
\[ \|f\|_2 = \|f - f_n\|_2 \leq \|f_0\|_2 \|f - f_n\|_2 \]
where $f$ is defined as in the earlier solution; and as before, this follows from Cauchy-Schwarz. Hence we can make $f$ arbitrarily close to 0, so it must be 0, hence $f \in E_0$.

(b) i. Let $f \in E$, consider the function $h$ defined by
\[ h = f - 2f(1)f_1. \] (27)
Clearly $h \in H$, $h(1) = f(1) - 2f(1)f_1(1) = f(1) - f(1) = 0$. We conclude that
\[ \forall f \in E, \exists h \in H, \; f = 2f(1)f_1 + h. \] (28)

To wit, $E$ is the vector space formed by the span of $f_1$ and $H$.

ii. Now, the mean of $f_1$ over $[0, 1]$ is zero, that is $f_1$ is in $E_0$. But (27) implies that in this case, $h$ also has zero-mean and thus $h \in H \cap E_0 = H_0$. In summary, if $f \in E_0$, (28) implies that $E_0$ is the vector space formed by the span of $f_1$ and $H_0$.

iii. Consider the sequence $u_n$ that matches $f_1$ on $[1/n, \; 1-1/n]$, and has a continuous linear extension on $[0, 1/n]$ and $[1-1/n, 1]$, given by
\[ u_n(t) = \begin{cases} \left(1 - \frac{t}{n}\right) t & \text{if } t \in [0, 1/n], \\ f_1(t) & \text{if } t \in [1/n, \; 1-1/n], \\ \left(1 - \frac{t}{n}\right) (t-1) & \text{if } t \in [1-1/n, 1]. \end{cases} \] (29)

The convergence of $u_n$ to $f_1$ is a consequence of the following simple computation
\[ \|f_1 - u_n\|_2 = \frac{1}{\sqrt{6n}}. \] (30)
Finally, we can check that by construction $u_n(1) = 0$ and $u_n$ is zero-mean, so $u_n \in H_0$. We conclude that the limit of $u_n$, $f_1$, is in $H_0$.

(c) Let $g \in H_0^\perp$. The idea is to hit $f_1 - u_n$ against $g$, and pass to the limit. We have
\[ \left| \int_0^1 f_1(t)g(t)dt \right| = \left| \int_0^1 (f_1(t) - u_n(t))g(t)dt \right| \leq \|f_1 - u_n\|_2 \|g\|_2 \] (31)
where we have used the fact that $\langle u_n, g \rangle = 0$ to get to the second term, and Cauchy-Schwarz yields the last inequality. Now because of (30), we conclude that
\[ \forall n \in \mathbb{N}, \; \int_0^1 f_1(t)g(t)dt \leq \frac{\|g\|_2}{\sqrt{6n}} \] (32)
in other words,
\[ \int_0^1 f_1(t)g(t)dt = 0. \] (33)
(d) Because $H_0$ is a subspace of $E_0$, we have $E_0^\perp \subset H_0^\perp$. Now let $g \in H_0^\perp$, and let $g \in E_0$. Because $E_0$ is the span of $f_1$ and $H_0$, there exist $\alpha \in \mathbb{C}$ and $h_0 \in H_0$ such that,

$$ g = \alpha f_1 + h_0. \quad (34) $$

Because $\langle g, h_0 \rangle = 0$ ($g \in H_0^\perp$) and $\langle g, f_1 \rangle = 0$ (see previous question), we conclude that

$$ \langle g, g \rangle = \langle g, \alpha f_1 + h_0 \rangle = \overline{\alpha} \langle g, f_1 \rangle + \langle g, h_0 \rangle = 0. \quad (35) $$

We conclude that $g \in E_0^\perp$, and thus $E_0^\perp = H_0^\perp$.

(e) We know that $E_0 = \{f_0\}^\perp$, so $E_0^\perp = \text{span}\{f_0\} = \text{span}\{f_0\}$, where all the orthogonal complements are taken in $E$. Therefore, $E_0^\perp$ contains only constant function.

Let $g \in H_0^\perp$, then $g \in H_0^\perp \subset H$, since $H_0^\perp$ is the orthogonal complement of $H_0$ in $H$, and therefore $g(1) = 0$. Now, $g \in E_0^\perp = H_0^\perp$, and thus $g$ is constant, so $g(t) = g(1) = 0$, and we conclude that $g = 0$.

In summary, $H_0^\perp = \{0\}$.

i. Alternative proof that $g \in E_0^\perp$ is constant. For shorthand, let $\mu = \langle g, f_0 \rangle$ be the mean, and $g_0 = g - \mu f_0$ as above. Then

$$ \|g_0\|^2 = \langle g - \mu f_0, g - \mu f_0 \rangle = \langle g, g - \mu f_0 \rangle - \langle \mu f_0, g - \mu f_0 \rangle = -\langle \mu f_0, g - \mu f_0 \rangle $$

since $\langle g, g - \mu f_0 \rangle = 0$ because $g - \mu f_0 \in E_0$ and $g \in E_0^\perp$. Thus

$$ \|g_0\|^2 = -\langle \mu f_0, g - \mu f_0 \rangle = |\mu|^2 \|f_0\|^2 - \mu \langle f_0, g \rangle = |\mu|^2 - |\mu|^2 = 0 $$

hence $g_0 = 0$, i.e., $g$ is a constant function (it doesn’t deviate from its mean).

ii. Second alternative proof that $g \in E_0^\perp$ is constant. We define $g_0$ to be the function $g$ centered around its mean,

$$ g_0 = g - \int_0^1 g(t) dt, = g - \langle g, f_0 \rangle f_0, \quad (36) $$

where we recall, that $f_0(t) = 1, \forall t \in [0, 1]$. By construction, $g_0 \in E_0$. Since $g \in E_0^\perp$, we have

$$ 0 = \langle g, g_0 \rangle = \langle g, g - \langle g, f_0 \rangle f_0 \rangle = \|g\|^2 - \|\langle g, f_0 \rangle f_0\|^2 = \|g\|^2 \|f_0\|^2 - \|\langle g, f_0 \rangle\|^2. \quad (37) $$

In short,

$$ \|g\|^2 \|f_0\|^2 = \|\langle g, f_0 \rangle\|^2. \quad (38) $$

We recognize the form of Cauchy-Schwartz, when the two vectors $g$ and $f_0$ are colinear. We conclude that $g$ is constant.

(f) The apparent contradiction stems from the fact that we work within a pre-Hilbert space, $H$. But $H$ is not complete, so it is not a Hilbert space. Despite the fact that $H_0^\perp = \{0\}$, we only have

$$ H_0 = \overline{H_0} \neq H. \quad (39) $$

For instance $f(t) = 1 - t$ is in $H$, but not in $H_0$. 

7