Applied Analysis Preliminary Exam
9:00 AM – 12:00 PM, August 17, 2021

Instructions You have three hours to complete this exam. Work all five problems; there are no optional problems. Each problem is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are directly proving such a theorem (when in doubt, ask the proctor). If you cannot finish part of a question, you may wish to move on to the next part; problems are graded with partial credit. **Write your student number on your exam, not your name**.

Problem 1 (20 points) The following two problems are unrelated.

(a) Let $\mathbf{r} \in \mathbb{R}^3$ and $\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{r^3}\mathbf{r}$ be a vector field where $r = \|\mathbf{r}\|_2$, and q and ϵ_0 are constants; you may recognize this as the electric field due to a charge q at the origin. By using the formula for the surface area of a sphere, observe that

$$\oiint_S \mathbf{E} \cdot \hat{\mathbf{n}}\, dS = \frac{q}{\epsilon_0}$$

if S is a sphere of radius $R > 0$ (no work necessary on your part). Show that in fact this equation holds for *any* bounded, smooth closed surface S that encloses the origin (this is a simple version of Gauss' law). *Hint: is* $\mathbf{E}$ *divergence-free in some regions?*

(b)

i. For every $n \in \mathbb{Z}$, show there is a unique solution $\alpha \in (-\pi/2, \pi/2]$ solving the equation $\cos(\alpha) = n^{\frac{3}{2}}\alpha$. *In the interest of time, you may give details for $n = 1$ and just a quick proof sketch for all other n.*

ii. Let α_n be the unique solution in $(-\pi/2, \pi/2]$ to the equation $\cos(\alpha) = n^{\frac{3}{2}}\alpha$. Define the function

$$f(x) = \sum_{n\in\mathbb{Z}} \alpha_n \cos(nx).$$

Prove f is a continuous function. *Hint: what do you know about growth/decay of α_n?*

Problem 2 (20 points) Let $F = C([0,1])$ be the Banach space of continuous real-valued functions on $[0,1]$ equipped with the sup-norm, $\|\cdot\|_\infty$, and let $G = C^1([0,1])$ be the Banach space of real-valued continuously differentiable functions on $[0,1]$ equipped with the norm

$$\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty. \tag{1}$$

(a) Prove that $\|f\|_{C^1}$ defined by (1) is a norm on G.

(b) We note that $G \subset F$, and we consider the canonical injection

$$\mathcal{I} : (G, \|\cdot\|_{C^1}) \longrightarrow (F, \|\cdot\|_\infty) \tag{2}$$
$$f \longmapsto f \tag{3}$$

Prove that $\mathcal{I}$ is a compact operator. *Hint: you might consider using the Arzelà-Ascoli theorem.*

Problem 3 (20 points) Let $F = C([0,1])$ be the Banach space of continuous real-valued functions on $[0,1]$ equipped with the sup-norm, $\|\cdot\|_\infty$. Let $u \in F$, and define the function Tu on $[0,1]$ by

$$Tu(x) = \begin{cases} \dfrac{1}{x}\displaystyle\int_0^x u(t)dt & \text{if } x \in (0,1], \\ u(0) & \text{if } x = 0. \end{cases} \tag{4}$$

(a) Prove that $Tu \in F$, and that

$$\forall u \in F, \quad \|Tu\| \leq \|u\|. \tag{5}$$

Conclude that T belongs to $\mathcal{B}(F)$.

(b) Prove that the point spectrum (eigenvalues) of T is equal to $(0,1]$. Determine the corresponding eigenfunctions.

(c) Prove that $\|T\|_{\mathcal{B}(F)} = 1$.

Problem 4 (20 points) Let X and Y be normed linear spaces, and $T : X \to Y$ a bounded linear transformation. Define the *transpose* $T^\top : Y^* \to X^*$ as the map that sends any $\phi \in Y^*$ to the linear functional $T^\top \phi \stackrel{\text{def}}{=} \phi \circ T \in X^*$, i.e., $(T^\top \phi)(x) = \phi(Tx) \, \forall x \in X$. Note that $T^\top$ is also a bounded linear transformation.

Let X and Y be Banach spaces, and suppose T is injective and has closed range. Prove that $T^\top$ is surjective. *Hint: you may wish to use major theorems, including the Hahn-Banach theorem.*

Problem 5 (20 points) Let E be the inner-product space of complex-valued continuous functions on $[0,1]$, equipped with the inner product

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}dt, \tag{6}$$

and the corresponding norm, $\|\cdot\|_2$. Let E_0 be the subspace of E composed of functions f with zero-mean,

$$\int_0^1 f(t) = 0. \tag{7}$$

We consider the following inner-product spaces H and H_0 defined respectively by

$$H = \{f \in E; f(1) = 0\}, \quad \text{and} \quad H_0 = E_0 \cap H = \left\{ f \in E; f(1) = \int_0^1 f(t)dt = 0 \right\}. \tag{8}$$

(a) Prove that H_0 is a proper closed subspace of H.

(b) Let $f_1 \in E_0$ defined by

$$\forall t \in [0,1], \quad f_1(t) = t - \frac{1}{2}. \tag{9}$$

i. Prove that E is equal to the vector space spanned by f_1 and H.

ii. Prove that E_0 is equal to the vector space spanned by f_1 and H_0.

iii. Prove that f_1 is an element of the closure, $\overline{H_0}$, of H_0 in E.
Hint: you might construct a sequence of functions $u_n \in H_0$, such that $\lim_{n\to\infty} \|f_1 - u_n\|_2 = 0$. *You need not provide an explicit formula for (u_n) as long as it is clear they have the desired property.*

(c) Let $H_0^\perp$ be the orthogonal complement of H_0 in H. Let $g \in H_0^\perp$, and let $u_n \in H_0$, such that $\lim_{n\to\infty} \|f_1 - u_n\|_2 = 0$. Prove that

$$\left| \int_0^1 f_1(t)\overline{g(t)}dt \right| \le \|f_1 - u_n\|_2 \, \|g\|_2. \tag{10}$$

Conclude that $\langle f_1, g \rangle = 0$.

(d) Show $H_0^\perp = E_0^\perp$ (where E_0 is the orthogonal complement of E_0 in E).

(e) Show that $E_0^\perp$ contains only constant functions, and then show $H_0^\perp = \{0\}$.

(f) Reconcile the apparent contradiction created by combining (a) and (e). To wit,

$$H_0^\perp \oplus H_0 = H_0 \neq H. \tag{11}$$