Problem 1 (20 points) The following two problems are unrelated.

(a) Let \( \mathbf{r} \in \mathbb{R}^3 \) and \( \mathbf{E}(\mathbf{r}) = q/4\pi\epsilon_0 \mathbf{r}/r^3 \) be a vector field where \( r = \| \mathbf{r} \|_2 \), and \( q \) and \( \epsilon_0 \) are constants; you may recognize this as the electric field due to a charge \( q \) at the origin. By using the formula for the surface area of a sphere, observe that
\[
\iint_S \mathbf{E} \cdot \mathbf{n} dS = \frac{q}{\epsilon_0}
\]
if \( S \) is a sphere of radius \( R > 0 \) (no work necessary on your part). Show that in fact this equation holds for any bounded, smooth closed surface \( S \) that encloses the origin (this is a simple version of Gauss’ law). Hint: is \( E \) divergence-free in some regions?

(b) i. For every \( n \in \mathbb{Z} \), show there is a unique solution \( \alpha \in (-\pi/2, \pi/2] \) solving the equation \( \cos(\alpha) = n/2 \). In the interest of time, you may give details for \( n = 1 \) and just a quick proof sketch for all other \( n \).

ii. Let \( \alpha_n \) be the unique solution in \( (-\pi/2, \pi/2] \) to the equation \( \cos(\alpha) = n/2 \). Define the function \( f(x) = \sum_{n \in \mathbb{Z}} \alpha_n \cos(nx) \).

Prove \( f \) is a continuous function. Hint: what do you know about growth/decay of \( \alpha_n \) ?

Problem 2 (20 points) Let \( F = C([0, 1]) \) be the Banach space of continuous real-valued functions on \([0, 1]\) equipped with the sup-norm, \( \| \cdot \|_\infty \), and let \( G = C^1([0, 1]) \) be the Banach space of real-valued continuously differentiable functions on \([0, 1]\) equipped with the norm
\[
\| f \|_{C^1} = \| f \|_\infty + \| f' \|_\infty.
\] (1)

(a) Prove that \( \| f \|_{C^1} \) defined by (1) is a norm on \( G \).

(b) We note that \( G \subset F \), and we consider the canonical injection
\[
\mathcal{I} : (G, \| \cdot \|_{C^1}) \longrightarrow (F, \| \cdot \|_\infty)
\] (2)
\[
f \longmapsto f
\] (3)

Prove that \( \mathcal{I} \) is a compact operator. Hint: you might consider using the Arzelà-Ascoli theorem.

Problem 3 (20 points) Let \( F = C([0, 1]) \) be the Banach space of continuous real-valued functions on \([0, 1]\) equipped with the sup-norm, \( \| \cdot \|_\infty \). Let \( u \in F \), and define the function \( Tu \) on \([0, 1]\) by
\[
Tu(x) = \begin{cases} 
\frac{1}{x} \int_0^x u(t) dt & \text{if } x \in (0, 1], \\
u(0) & \text{if } x = 0.
\end{cases}
\] (4)

(a) Prove that \( Tu \in F \), and that
\[
\forall u \in F, \quad \| Tu \| \leq \| u \|.
\] (5)

Conclude that \( T \) belongs to \( B(F) \).
(b) Prove that the point spectrum (eigenvalues) of \( T \) is equal to \((0,1]\). Determine the corresponding eigenfunctions.

(c) Prove that \( \|T\|_{B(F)} = 1 \).

**Problem 4 (20 points)** Let \( X \) and \( Y \) be normed linear spaces, and \( T : X \to Y \) a bounded linear transformation. Define the transpose \( T^\top : Y^* \to X^* \) as the map that sends any \( \phi \in Y^* \) to the linear functional \( T^\top \phi \defeq \phi \circ T \in X^* \), i.e., \( (T^\top \phi)(x) = \phi(Tx) \forall x \in X \). Note that \( T^\top \) is also a bounded linear transformation. Let \( X \) and \( Y \) be Banach spaces, and suppose \( T \) is injective and has closed range. Prove that \( T^\top \) is surjective. *Hint: you may wish to use major theorems, including the Hahn-Banach theorem.*

**Problem 5 (20 points)** Let \( E \) be the inner-product space of complex-valued continuous functions on \([0,1]\), equipped with the inner product
\[
\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}\,dt,
\]
and the corresponding norm, \( \| \cdot \|_2 \). Let \( E_0 \) be the subspace of \( E \) composed of functions \( f \) with zero-mean,
\[
\int_0^1 f(t) = 0.
\]
We consider the following inner-product spaces \( H \) and \( H_0 \) defined respectively by
\[
H = \{ f \in E; f(1) = 0 \}, \quad \text{and} \quad H_0 = E_0 \cap H = \left\{ f \in E; f(1) = \int_0^1 f(t)dt = 0 \right\}.
\]

(a) Prove that \( H_0 \) is a proper closed subspace of \( H \).

(b) Let \( f_1 \in E_0 \) defined by
\[
\forall t \in [0,1], \quad f_1(t) = t - \frac{1}{2}.
\]

i. Prove that \( E \) is equal to the vector space spanned by \( f_1 \) and \( H \).

ii. Prove that \( E_0 \) is equal to the vector space spanned by \( f_1 \) and \( H_0 \).

iii. Prove that \( f_1 \) is an element of the closure, \( \overline{H_0} \), of \( H_0 \) in \( E \).

*Hint: you might construct a sequence of functions \( u_n \in H_0 \), such that \( \lim_{n \to \infty} \|f_1 - u_n\|_2 = 0 \). You need not provide an explicit formula for \( (u_n) \) as long as it is clear they have the desired property.*

(c) Let \( H_0^\perp \) be the orthogonal complement of \( H_0 \) in \( H \). Let \( g \in H_0^\perp \), and let \( u_n \in H_0 \), such that \( \lim_{n \to \infty} \|f_1 - u_n\|_2 = 0 \). Prove that
\[
\left| \int_0^1 f_1(t)\overline{g(t)}\,dt \right| \leq \|f_1 - u_n\|_2 \|g\|_2.
\]

Conclude that \( \langle f_1, g \rangle = 0 \).

(d) Show \( H_0^\perp = E_0^\perp \) (where \( E_0 \) is the orthogonal complement of \( E_0 \) in \( E \)).

(e) Show that \( E_0^\perp \) contains only constant functions, and then show \( H_0^\perp = \{0\} \).

(f) Reconcile the apparent contradiction created by combining (a) and (e). To wit,
\[
H_0^\perp \oplus H_0 = H_0 \neq H.
\]