Applied Analysis Preliminary Exam

9:00-12:00 August 16, 2022

Instructions: You have three hours to complete this exam. Work all five problems; each is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are being asked to prove such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name.

Problem 1: Metric Spaces

(a) Suppose that (X, d_X) and (Y, d_Y) are metric spaces. Show that $X \times Y$ is a metric space, with an appropriate metric.

(b) Let $A = \{(\xi_1, \xi_2, \ldots) : \xi_j \in \mathbb{R}\}$ be the space of real sequences and $\mu_j > 0$. Show that

$$d(\xi, \eta) = \sum_{j=1}^{\infty} \mu_j \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|}$$

is a metric on A whenever $\sum_{j=1}^{\infty} \mu_j$ converges.

(c) Let $M \subset \ell^\infty$ be the subspace consisting of all sequences (ξ_j) with at most finitely many nonzero terms. Show that M is not complete.

Problem 2: Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = \frac{x^n y^m}{x^2 + y^2}$$

for $(x, y) \neq (0, 0)$ and $m, n \in \mathbb{N}$. Show that $\lim_{(x,y)\to(0,0)} f(x, y)$ exists if and only if $m + n > 2$.

Problem 3: Let H be an infinite dimensional Hilbert space and $K \colon H \to H$ a compact linear operator. Prove the following statements.

(a) $0 \in \sigma(K)$, where $\sigma(K)$ is the spectrum of K.

(b) $\mathrm{Ker}(I - K) = \{0\}$ *iff* $\mathrm{Range}(I - K) = H$.

(c) $\sigma(K) = \sigma_p(K) \cup \{0\}$, where $\sigma_p(K)$ is the point spectrum of K.

Problem 4: Let $f(t)$ be a complex-valued function defined for $t \geq 0$. Its **Laplace Transform** Lf is a function defined on $s > 0$ given by

$$g(s) = (Lf)(s) = \int_0^\infty f(t) e^{-st}\, dt.$$

(a) Prove that L is a bounded linear map from $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$;

(b) Prove that

$$\|L\| \leq \sqrt{\pi}.$$

Hint: multiply and divide the integrand by $t^{1/4}$ to aid in integration.

Please turn over

Problem 5: Let $A : H \to H$ be a compact and symmetric operator defined on a Hilbert space H. Define the *Rayleigh quotient* $R_A(x)$ as follows:

$$R_A(x) = \frac{(Ax, x)}{\|x\|^2}.$$

Denote the positive eigenvalues of A, indexed in decreasing order, by λ_k with $k = 1, 2, \dots,$ with corresponding eigenvectors z_n. Recall that we can compute

$$\lambda_N = \max_{x \perp \{z_1, z_2, \dots, z_{N-1}\}} \frac{(Ax, x)}{\|x\|^2}. \tag{1}$$

and the maximum is achieved by z_N.

(a) Prove that

$$\lambda_N = \max_{S_N} \left(\min_{x \in S_N} R_A(x) \right)$$

where S_N is any N-dimensional vector subspace of H.

(b) Prove that

$$\lambda_N = \min_{S_{N-1}} \left(\max_{x \perp \{S_{N-1}\}} R_A(x) \right)$$

where S_{N-1} is as in (a).