Time allowed: 3 hours.

3 pages (including this one).

Instructions:

- Work all five problems; each is worth 20 points; please start each problem on a new page.
- Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page).
- You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are being asked to prove such a theorem (when in doubt, ask the proctor).
- Write your **student number** on your exam, not your name.
- Remote participants will be required to take photos or scan their written work which will be sent to Laura (amgradco@colorado.edu) no later than 10 minutes after the end of the exam (i.e., by 4:40pm).

Problem 1. [20 = 5 + 15]
Let f be a real-valued function.

A. In this question we assume that f is a continuously differentiable function, $f \in C^1([0,1])$. Prove that
$$\lim_{n\to\infty} \quad n\int_0^1 e^{-nx} f(x)dx = f(0). \tag{1}$$

B. In this question, we relax our hypothesis and only assume that f is continuous, $f \in C([0,1])$. Prove that (1) still holds. *Hint:* use the Weierstrass approximation theorem.

Problem 2. [20]
Prove that the topological dual space of $\ell^\infty(\mathbb{N})$ is not isomorphic to $\ell^1(\mathbb{N})$ under the standard isomorphism; i.e., show that not all dual elements φ can be written as (for $x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty$) $\varphi(x) = \sum_{n\in\mathbb{N}} x_n y_n$ for some $y = (y_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$. *Hint:* Use the Hahn-Banach theorem or one of its corollaries.

Problem 3. [20 = 5+10+5]
Let X and Y be Banach spaces.

A. Let $T : X \to Y$ be a bounded linear operator. Prove that T maps weakly convergent sequence to weakly convergent sequences, i.e., $x_n \rightharpoonup x \implies Tx_n \rightharpoonup Tx$.

B. Let $T : X \to Y$ be a *compact* linear operator. Prove that T maps weakly convergent sequences to strongly convergent sequences. *Note:* You may use the fact that weakly convergent sequences are bounded (a consequence of the Uniform Boundedness theorem).

C. Conversely, prove that if X is reflexive, then if a bounded linear operator $T : X \to Y$ maps weakly convergent sequences to strongly convergent sequences, then T must be compact.

Problem 4. [20 = 4 × 5]
Let $C([0,1])$ be the Banach space of complex-valued continuous functions on $[0,1]$ equipped with the norm,

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|. \tag{2}$$

$C([0,1])$ is also a pre-Hilbert subspace of the Hilbert space $L^2([0,1])$, equipped with the inner product

$$\langle f, g\rangle = \int_{[0,1]} f(x)\overline{g(x)}dx, \tag{3}$$

and the associated norm

$$\|f\|_2 = \left[\int_{[0,1]} |f(x)|^2 dx\right]^{1/2}. \tag{4}$$

We consider a subspace E of $C([0,1])$; E is also a subspace of $L^2([0,1])$. We assume that E is closed in $L^2([0,1])$ for the topology induced by the $\|\cdot\|_2$ norm, defined by (4).

A. Prove that E is closed in $C([0,1])$ for the topology induced by the $\|\cdot\|_\infty$ norm, defined by (2).

B. Prove that there exists a constant $\gamma > 0$ such that

$$\forall f \in E, \quad \|f\|_2 \leq \|f\|_\infty \leq \gamma\|f\|_2. \tag{5}$$

Hint: use the open mapping theorem.

In what follows we will prove that E has finite dimension, and we will bound $\dim E$. We reason by contradiction: we assume that the dimension of E is infinite. In this case we can exhibit an orthonormal basis, $(e_n)_{n\geq 1}$, of E, equipped with the inner product (3).

C. Let $k \geq 1$ be any fixed integer. Prove that for any choice of $\alpha_1, \ldots, \alpha_k$ in $\mathbb{C}$ we have,

$$\forall x \in [0,1], \quad \left|\sum_{i=1}^k \alpha_i e_i(x)\right| \leq \gamma \left(\sum_{i=1}^k |\alpha_i|^2\right)^{1/2}. \tag{6}$$

D. Deduce that

$$\forall x \in [0,1], \quad \sum_{i=1}^k |e_i(x)|^2 \leq \gamma^2. \tag{7}$$

E. Conclude that the dimension of E is finite and less than γ^2.

Problem 5. [20 = 2 + 2 + 6 + 2 + 2 + 2 + 4]
This problem is decomposed into ***very elementary questions****. If you cannot answer one question, please skip it and simply assume its result to hold. As often as not the answer to question n is given by the result proved in question n − 1.*

We recall the following set notation. If A and B are two subsets of X, the difference between A and B is the set of all points of X that belong to A but do not belong to B,

$$A - B = \{x \in X; x \in A \text{ and } x \notin B\}. \tag{8}$$

Let (X, μ) be a measurable space. We assume that the measure μ is finite. Let φ_n be a sequence of real-valued measurable functions $\varphi_n : X \to \mathbb{R}$. We assume that the sequence $\{\varphi_n\}$ converges almost everywhere to a finite measurable real-valued function $\varphi : X \to \mathbb{R}$. Therefore, by removing from X, if necessary, a set of measure zero, we may assume – *everywhere in this problem* – that

$$\forall x \in X, \qquad \varphi_n(x) \text{ is finite, and } \lim_{n\to\infty} \varphi_n(x) = \varphi(x).$$

Let k, m be two positive integers. We define the following measurable set,

$$B_k(m) = \left\{x \in X;\ \forall n \geq k,\ |\varphi_n(x) - \varphi(x)| < \frac{1}{m}\right\}. \tag{9}$$

A. Prove that $B_k(m+1) \subset B_k(m) \subset B_{k+1}(m)$.

B. Prove that for all positive integers m,

$$X = \bigcup_{k=1}^{\infty} B_k(m). \tag{10}$$

C. Prove that for all positive integers m,

$$\mu(X) = \lim_{k\to\infty} \mu(B_k(m)). \tag{11}$$

Hint: express $\bigcup_{k=1}^{\infty} B_k(m)$ as a countable union of measurable *mutually disjoint* sets, and observe that the resulting sum of measures is a telescopic series.

D. Let $\varepsilon > 0$. Prove that there exists a sequence of integers $\{k_m\}_{m\geq 1}$, such that

$$\mu(X) - \mu(B_{k_m}(m)) < \frac{\varepsilon}{2^m}. \tag{12}$$

E. Let $\varepsilon > 0$. We define

$$Y_\varepsilon = \bigcap_{m=1}^{\infty} B_{k_m}(m), \tag{13}$$

where the $\{k_m\}$ is the sequence defined in the previous question. Prove that Y_ε is measurable.

F. Prove that $\mu(X - Y_\varepsilon) < \varepsilon$, where the set difference $X - Y_\varepsilon$ is defined in (8).

G. Conclude that

$$\forall \varepsilon > 0,\ \exists Y_\varepsilon, \quad \text{such that } \mu(X - Y_\varepsilon) < \varepsilon, \text{ and } \varphi_n \text{ converges uniformly to } \varphi \text{ on } Y_\varepsilon. \tag{14}$$

Solution 1.
1. Define
$$I_n(f) = n\int_0^1 e^{-nx} f(x)dx. \tag{15}$$
Using integration by part,
$$I_n(f) = -e^{-nx}f(x)\Big|_0^1 + \int_0^1 e^{-nx}f'(x)dx. \tag{16}$$
$f'(x)$ is continuous on the compact $[0,1]$ and therefore $\exists K > 0, \forall x \in [0,1]$, $\left|f'(x)\right| \le K$, and therefore
$$\left|\int_0^1 e^{-nx}f'(x)dx\right| \le K\int_0^1 e^{-nx}dx = \frac{K(1-e^{-n})}{n} \longrightarrow 0 \quad \text{as } n \to \infty. \tag{17}$$
We conclude
$$\lim_{n\to\infty} I_n(f) = f(0). \tag{18}$$
2. Let $f \in C([0,1])$, and let $\varepsilon > 0$. Because the set of polynomials is dense in $C([01,])$ for the topology associated with the $\|\cdot\|_\infty$ norm, there exits a polynomial $p(x)$, such that
$$\|f - p\|_\infty < \varepsilon/3. \tag{19}$$
Now, $p \in C^1([0,1])$, and from the previous question we know that
$$\exists N_0, \quad \forall n \ge N_0, \quad \left|I_n(p) - p(0)\right| < \varepsilon/3. \tag{20}$$
Now, because of (19), we have
$$\left|f(0) - p(0)\right| < \varepsilon/3. \tag{21}$$
Putting everything together, we get for all $n \ge N_0$,
$$\left|I_n(f) - f(0)\right| \le \left|I_n(f) - I_n(p)\right| + \left|I_n(p) - p(0)\right| + \left|p(0) - f(0)\right| \tag{22}$$
$$\le \left|I_n(f-p)\right| + \frac{2}{3}\varepsilon \tag{23}$$
$$\le \|f-p\|_\infty \int_0^1 ne^{-nx}dx + \frac{2}{3}\varepsilon \tag{24}$$
$$\le \varepsilon \tag{25}$$
We conclude that
$$\lim_{n\to\infty} I_n(f) = f(0), \tag{26}$$
as advertised.

Solution 2.
We'll write ℓ^∞ for $\ell^\infty(\mathbb{N})$ (and similarly for ℓ^1), and $(\ell^\infty)^*$ for the topological dual (here, topological is just in contrast to *algebraic* dual; both dual spaces are all linear functionals, but the topological dual also requires them to be continuous, i.e., bounded).

Below are two different proofs.

First proof Let $c_0 = c_0(\mathbb{N})$ be the subspace of ℓ^∞ consisting of convergent sequences; this is a subspace because of the additive and scaling properties of limits. Then for $x \in c_0$, define $\phi(x) = \lim_n x_n$, which is a bounded linear functional. Thus by **Hahn-Banach**, it can be extended to some $\varphi \in (\ell^\infty)^*$. This φ cannot be represented by anything in ℓ^1; if we could write $\varphi(x) = \sum_{i=1}^\infty x_i y_i$ for some $y \in \ell^1$, then by using $x = e_i = (0, 0, \ldots, 0, 1, 0, \ldots)$ (the canonical basis vector) we know $\varphi(e_i) = 0$ since $e_i \in c_0$ so we can use the limit definition; yet this implies the i^{th} element of y is zero, and this is true for all $i \in \mathbb{N}$, hence $y = 0$. Yet this cannot be right, since if we take $x = (1, 1, 1, \ldots)$, then $\varphi(x) = 1$ yet if $\varphi(x) = \sum_{i=1}^\infty x_i y_i$ we get $\varphi(x) = 0$ since $y = 0$.

Note that this proof doesn't show the two spaces cannot be isomorphic, only that they cannot be isomorphic under the suggested isomorphism.

Second proof (probably less popular, since this corollary of Hahn-Banach isn't in Hunter-Nachtergaele). We'll prove that if X is a normed linear space and X^* is separable, then X must be separable. This implies the result, because ℓ^1 is separable, and if it is isomorphic to $(\ell^\infty)^*$, then $(\ell^\infty)^*$ is also separable, hence we conclude ℓ^∞ is separable, but it is well-known that this is not true (e.g., for proof, you can make a bijection between ℓ^∞ and either the binary expansion of the interval $[0, 1]$ or the powerset of $\mathbb{N}$).

So to prove our statement, we assume X^* is separable, and hence the unit sphere S in X^* is separable, so let $(\varphi_n)_{n\in\mathbb{N}}$ be a countable dense set of S. Since $\|\varphi_n\| = 1$, we can find some $x_n \in X$ with $\|x_n\| = 1$ and $|\varphi_n(x_n)|$ arbitrarily close to $\|\varphi_n\|$; in particular, select such an x_n such that $|\varphi_n(x_n)| \geq 1/2$. Let $V = \operatorname{span}(x_1, x_2, \ldots)$. Then V is separable itself, since the *rational* span of $(x_1, x_2, \ldots)$ is dense in V (this is a standard argument). Now we must show that V is dense in X, which would imply X is separable.

Suppose V is not dense in X, so there exists $\epsilon > 0$ and $x \in X$ such that there is no x_n within ϵ of x. Now, by a well-known corollary of **Hahn-Banach**, there exists $\varphi \in X^*$ with $\|\varphi\| = 1$ and $\varphi(x_n) = 0$.[1] But by density of φ_n in the unit sphere of X^*, we can find some $n \in \mathbb{N}$ with $\|\varphi - \varphi_n\|$ arbitrarily small. Yet

$$(\forall n \in \mathbb{N}) \quad \|\varphi - \varphi_n\| = \sup_{\|x'\|=1} |\varphi(x') - \varphi_n(x')\| \geq |\varphi(x_n) - \varphi_n(x_n)| = |\varphi_n(x_n)| \geq \frac{1}{2}$$

which contradicts that $\|\varphi - \varphi_n\|$ can be arbitrarily small.

[1] e.g., see Lemma 4.6-7 "Existence of a functional" in Kreyszig's book: let V be a proper closed subspace of X, and $x \in X \setminus V$ with $\epsilon = \inf_{v\in V} \|v - x\|$, then $\exists \varphi \in X^*$ with $\|\varphi\| = 1$, $\varphi(v) = 0\,(\forall v \in V)$, and $\varphi(x) = \epsilon$. To see this from the Hahn-Banach theorem, let $\widetilde{V} = \operatorname{span}(V, x)$ and define $\phi(v + \alpha x) = \alpha\epsilon$, and then extend ϕ to φ on the whole space by invoking Hahn-Banach.

Solution 3.

A. First, we show weak convergence of (Tx_n), i.e., $Tx_n \rightharpoonup Tx$. Let $\psi \in Y^*$. Then

$$\lim_{n\to\infty} \psi(Tx_n) = \lim_{n\to\infty} (\psi \circ T)(x_n) = (\psi \circ T)(x) = \psi(Tx)$$

where we used in the second inequality that $(\psi \circ T) \in X^*$ because $\psi \in Y^*$ and T bounded implies $(\psi \circ T)$ is bounded (and also clearly linear), and the third inequality follows from $x_n \rightharpoonup x$ (and the definition of weak convergence).

B. Let $T : X \to Y$ be a compact (linear) operator, and (x_n) a weakly convergent sequence in X. We wish to prove (Tx_n) converges strongly.

Let $x_n \rightharpoonup x$. If (Tx_n) does converge, we can guess that it will converge to Tx, so we will attempt to prove this.

Now, argue by contradiction: suppose (Tx_n) does not converge to Tx. This means $\exists \epsilon > 0$ such that there is a subsequence (n_k) with

$$(\forall k \in \mathbb{N}) \quad \|Tx_{n_k} - Tx\| \geq \epsilon. \tag{27}$$

Finally, using the hint (that (x_n) is bounded), and the compactness, then since x_{n_k} is a bounded sequence, (Tx_{n_k}) must have a convergent subsequence (i.e., the set $\{Tx_{n_k}\}$ is precompact). Let y denote the limit of this convergent subsequence. But this convergent subsequence (which we won't explicitly name, to avoid ugly notation) is still a subsequence of (Tx_n) and therefore weakly converges to Tx. Since it strongly converges to y, then it also weakly converges to y. Since limits (weak or strong) are unique, then $y = Tx$ [or, one can also deduce this by saying that if a sequence converges strongly and weakly, it must be to the same limit]. But this contradicts Eq. (27).

C. We wish to show T is compact. This means that if A is bounded, then $T(A)$ is precompact (aka relatively compact).

First, we prove another useful characterization of compact operators (in fact, this is stated in the Hunter & Nachtergaele book below Definition 5.42 without proof, so students got full credit for stating this as the "definition" without proving it):

> A linear operator T is compact iff the image of any bounded sequence (x_n) has a convergent subsequence.

One direction of the "iff" is easy and not relevant to our problem. For the other direction (also very easy, almost a tautology), supposing the image of any bounded sequence (x_n) has a convergent subsequence, we want to show T is compact. Let A be an bounded set, and let $(y_n) \subset T(A)$ be any subsequence in $T(A)$. We wish to show $T(A)$ is precompact, i.e., show there is a convergent subsequence of (y_n). Since $(y_n) \subset T(A)$, we can write each y_n as $y_n = T(x_n)$ for some $x_n \in A$. Since A is bounded, then (x_n) is bounded. Then by assumption, there is a convergent subsequence of the image of (x_n), hence (y_n) has a convergent subsequence.

Back to the problem, we wish to show T is compact, and we will do so by showing that the image of any bounded sequence (x_n) has a convergent subsequence. If (x_n) is a bounded sequence, then because X is reflexive, the Banach-Alouglu theorem proves that (x_n) is weakly precompact, meaning that there is a weakly convergent subsequence[2]. But then the assumption that T maps weakly convergent sequences to strongly convergent ones immediately implies that $T(x_n)$ has a convergent subsequence, and this shows that T is compact.

[2]More precisely, the Banach-Alouglu theorem says the closed unit ball in a dual space Z^* is weak-* compact; we identify $Z = X^*$ and use reflexivity (so $Z^* = X$) to get weak-compactness, and for moving from weak-compactness to weak-precompactness, we take advantage that the sequential and topological notions of weak closure coincide in a Banach space, due to the Eberlein-Šmulian theorem, which is not obvious since in general the weak topology is not metrizable.

Solution 4.

A. Consider the canonical injection

$$A : C([0,1]) \longrightarrow L^2([0,1]) \tag{28}$$
$$f \longmapsto f \tag{29}$$

A is continuous,

$$\|f\|_2^2 = \int_0^1 |f(x)|^2 dx \leq \|f\|_\infty^2. \tag{30}$$

The image of $E \subset C([0,1])$ by A can be identified with $E \subset L^2([0,1])$, which is closed in $L^2([0,1])$. We conclude that E is closed in $C([0,1])$ for the topology induced by the $\|\cdot\|_\infty$ norm, since it is the pre-image of a closed set (E) by a continuous map (A). Alternatively, one can prove directly that E contains all its limit points (the actual details of a proof via this approach are of course basically the same)

B. The restriction of A to E, A_E is one-to-one and onto. Furthermore A is continuous (see (30)). By the open mapping theorem the inverse A_E^{-1} of A_E is continuous, and therefore there exists $\gamma > 0$, such that

$$\|f\|_\infty \leq \gamma \|f\|_2. \tag{31}$$

The other inequality is given by (30).

C. Let $(\alpha_1, \ldots, \alpha_k) \in \mathbb{C}^k$, we consider the vector f of E whose coordinates in the basis $e_i(x)$ are the $\{\alpha_i\}$,

$$f(x) = \sum_{i=1}^k \alpha_i e_i(x). \tag{32}$$

Using Parseval, we have

$$\|f\|_2 = \left(\sum_{i=1}^k |\alpha_i|^2\right)^{1/2}. \tag{33}$$

Combining (31) and (33), we conclude that for all $x \in [0,1]$,

$$\left|\sum_{i=1}^k \alpha_i e_i(x)\right| \leq \|f\|_\infty \leq \gamma \|f\|_2 = \gamma \left(\sum_{i=1}^k |\alpha_i|^2\right)^{1/2}. \tag{34}$$

D. For the choice $\alpha_i = \overline{e_i(x)}$, (6) yields

$$\forall x \in [0,1], \quad \sum_{i=1}^k |e_i(x)|^2 \leq \gamma \left(\sum_{i=1}^k |e_i(x)|^2\right)^{1/2}, \tag{35}$$

or equivalently

$$\forall x \in [0,1], \quad \sum_{i=1}^k |e_i(x)|^2 \leq \gamma^2. \tag{36}$$

E. Integrating (36) over the interval yields the advertised result. The dimension of E is bounded in terms of γ, which measures how tight is the bi-Lipschitz embedding (5).

Solution 5.

A. If $x \in B_k(m+1)$ then
$$\forall n \geq k, \quad |\varphi_n(x) - \varphi(x)| < \frac{1}{m+1} < \frac{1}{m}, \tag{37}$$
thus $x \in B_k(m)$. Also, if $x \in B_k(m)$ then
$$\forall n \geq k, \quad |\varphi_n(x) - \varphi(x)| < \frac{1}{m} \tag{38}$$
and therefore
$$\forall n \geq k+1, \quad |\varphi_n(x) - \varphi(x)| < \frac{1}{m} \tag{39}$$
thus $x \in B_{k+1}(m)$.

B. Let $m \geq 1$ be an integer. Clearly,
$$\bigcup_{k=1}^{\infty} B_k(m) \subset X. \tag{40}$$
Now, consider $x \in X$, because $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$, we have
$$\forall m \geq 1, \exists k,\ \forall n \geq k, \quad |\varphi_n(x) - \varphi(x)| < \frac{1}{m} \tag{41}$$
and thus $\exists k, x \in B_k(m)$.

C. Because $B_{k-1}(m) \subset B_k(m)$, we can consider the measurable set
$$C_k = B_k(m) - B_{k-1}(m) \quad \text{for } k \geq 2, \tag{42}$$
with
$$C_1 = B_1(m). \tag{43}$$
The $\{C_k\}_{k\geq 1}$ form a countable family of measurable mutually disjoint sets. So
$$\mu\left(\bigcup_{k\geq 1} C_k\right) = \sum_{k\geq 1} \mu(C_k), \tag{44}$$
where the sum on the right is bounded by $\mu(X)$. By definition of C_k,
$$\mu(C_k) = \mu(B_k(m)) - \mu(B_{k-1}(m)) \quad \text{and} \quad \mu(C_1) = \mu(B_1(m)). \tag{45}$$
Also,
$$\bigcup_{k\geq 1} C_k = \bigcup_{k\geq 1} B_k(m) = X \tag{46}$$
We conclude from (45) that $\sum_{k\geq 1} \mu(C_k)$ is a telescopic series, and thus
$$\mu(X) = \mu(B_1(m)) + \big(\mu(B_2(m)) - \mu(B_1(m))\big) + \ldots = \lim_{k\to\infty} \mu(B_k(m)). \tag{47}$$

D. Let $\varepsilon > 0$ and $m \geq 1$. Define $\eta = \varepsilon/2^m$. Because the sequence $\mu(B_k(m))$ is monotonically increasing and converges to $\mu(X)$, there exists k_η such that

$$\mu(X) - \mu(B_{k_\eta}(m)) \leq \eta = \frac{\varepsilon}{2^m}. \tag{48}$$

We define the term m of the sequence to be $k_m = k_\eta$. Note that k_m implicitly depends on ε, as illustrated in the next question.

E. The set $Y_\varepsilon = \bigcap_{m=1}^{\infty} B_{k_m}(m)$ is measurable: it is a countable intersection of measurable sets.

F. Now $Y_\varepsilon \subset X$ and

$$X - Y_\varepsilon = X - \bigcap_{m=1}^{\infty} B_{k_m}(m) = \bigcup_{m=1}^{\infty} \left\{X - B_{k_m}(m)\right\}. \tag{49}$$

So

$$\mu\left(X - Y_\varepsilon\right) \leq \sum_{m=1}^{\infty} \mu\left\{X - B_{k_m}(m)\right\}, \tag{50}$$

where the right-hand side converges since $\mu\left\{X - B_{k_m}(m)\right\} = \mu(X) - \mu(B_{k_m}(m)) \leq \varepsilon/2^m$. We conclude that

$$\mu\left(X - Y_\varepsilon\right) \leq \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = \varepsilon. \tag{51}$$

G. Let $\varepsilon > 0$, take $Y_\varepsilon = \bigcap_{m=1}^{\infty} B_{k_m}(m)$ constructed in question 5. Let $x \in Y_\varepsilon$, then

$$\forall m \geq 1, x \in B_{k_m}(m), \tag{52}$$

which means

$$\forall m \geq 1, \exists k_m, \forall n \geq k_m, \quad |\varphi_n(x) - \varphi(x)| < \frac{1}{m}. \tag{53}$$

Since the last statement is true for all $x \in Y_\varepsilon$, we conclude that

$$\forall m \geq 1, \exists k_m,\ \forall n \geq k_m, \quad \forall x \in Y_\varepsilon,\ |\varphi_n(x) - \varphi(x)| < \frac{1}{m}. \tag{54}$$

This statement means that φ_n converges uniformly to φ on Y_ε.

We have proved that if the sequence φ_n converges almost everywhere to φ on X, then it converges uniformly on a set whose measure is arbitrarily close to that of X.