Time allowed: 3 hours.

3 pages (including this one).

Instructions:

- Work all five problems; each is worth 20 points; please start each problem on a new page.
- Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page).
- You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are being asked to prove such a theorem (when in doubt, ask the proctor).
- Write your **student number** on your exam, not your name.
- Remote participants will be required to take photos or scan their written work which will be sent to Laura (amgradco@colorado.edu) no later than 10 minutes after the end of the exam (i.e., by 1:10pm).

Problem 1. [20 = 5 + 15]
Let f be a real-valued function.

A. In this question we assume that f is a continuously differentiable function, $f \in C^1([0,1])$. Prove that

$$\lim_{n\to\infty} \quad n\int_0^1 e^{-nx} f(x)dx = f(0). \tag{1}$$

B. In this question, we relax our hypothesis and only assume that f is continuous, $f \in C([0,1])$. Prove that (1) still holds. *Hint:* use the Weierstrass approximation theorem.

Problem 2. [20]
Prove that the topological dual space of $\ell^\infty(\mathbb{N})$ is not isomorphic to $\ell^1(\mathbb{N})$ under the standard isomorphism; i.e., show that not all dual elements φ can be written as (for $x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty$) $\varphi(x) = \sum_{n\in\mathbb{N}} x_n y_n$ for some $y = (y_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$. *Hint:* Use the Hahn-Banach theorem or one of its corollaries.

Problem 3. [20 = 5+10+5]
Let X and Y be Banach spaces.

A. Let $T : X \to Y$ be a bounded linear operator. Prove that T maps weakly convergent sequence to weakly convergent sequences, i.e., $x_n \rightharpoonup x \implies Tx_n \rightharpoonup Tx$.

B. Let $T : X \to Y$ be a *compact* linear operator. Prove that T maps weakly convergent sequences to strongly convergent sequences. *Note:* You may use the fact that weakly convergent sequences are bounded (a consequence of the Uniform Boundedness theorem).

C. Conversely, prove that if X is reflexive, then if a bounded linear operator $T : X \to Y$ maps weakly convergent sequences to strongly convergent sequences, then T must be compact.

Problem 4. [20 = 4 × 5]
Let $C([0,1])$ be the Banach space of complex-valued continuous functions on $[0,1]$ equipped with the norm,

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|. \tag{2}$$

$C([0,1])$ is also a pre-Hilbert subspace of the Hilbert space $L^2([0,1])$, equipped with the inner product

$$\langle f, g\rangle = \int_{[0,1]} f(x)\overline{g(x)}dx, \tag{3}$$

and the associated norm

$$\|f\|_2 = \left[\int_{[0,1]} |f(x)|^2 dx\right]^{1/2}. \tag{4}$$

We consider a subspace E of $C([0,1])$; E is also a subspace of $L^2([0,1])$. We assume that E is closed in $L^2([0,1])$ for the topology induced by the $\|\cdot\|_2$ norm, defined by (4).

A. Prove that E is closed in $C([0,1])$ for the topology induced by the $\|\cdot\|_\infty$ norm, defined by (2).

B. Prove that there exists a constant $\gamma > 0$ such that

$$\forall f \in E, \quad \|f\|_2 \le \|f\|_\infty \le \gamma \|f\|_2. \tag{5}$$

Hint: use the open mapping theorem.

In what follows we will prove that E has finite dimension, and we will bound $\dim E$. We reason by contradiction: we assume that the dimension of E is infinite. In this case we can exhibit an orthonormal basis, $(e_n)_{n\ge 1}$, of E, equipped with the inner product (3).

C. Let $k \ge 1$ be any fixed integer. Prove that for any choice of $\alpha_1, \dots, \alpha_k$ in $\mathbb{C}$ we have,

$$\forall x \in [0,1], \quad \left|\sum_{i=1}^{k} \alpha_i e_i(x)\right| \le \gamma \left(\sum_{i=1}^{k} |\alpha_i|^2\right)^{1/2}. \tag{6}$$

D. Deduce that

$$\forall x \in [0,1], \quad \sum_{i=1}^{k} |e_i(x)|^2 \le \gamma^2. \tag{7}$$

E. Conclude that the dimension of E is finite and less than γ^2.

Problem 5. [20 = 2 + 2 + 6 + 2 + 2 + 2 + 4]
*This problem is decomposed into **very elementary questions**. If you cannot answer one question, please skip it and simply assume its result to hold. As often as not the answer to question n is given by the result proved in question $n-1$.*

We recall the following set notation. If A and B are two subsets of X, the difference between A and B is the set of all points of X that belong to A but do not belong to B,

$$A - B = \{x \in X; x \in A \text{ and } x \notin B\}. \tag{8}$$

Let (X, μ) be a measurable space. We assume that the measure μ is finite. Let φ_n be a sequence of real-valued measurable functions $\varphi_n : X \to \mathbb{R}$. We assume that the sequence $\{\varphi_n\}$ converges almost everywhere to a finite measurable real-valued function $\varphi : X \to \mathbb{R}$. Therefore, by removing from X, if necessary, a set of measure zero, we may assume – *everywhere in this problem* – that

$$\forall x \in X, \qquad \varphi_n(x) \text{ is finite, and } \lim_{n\to\infty} \varphi_n(x) = \varphi(x).$$

Let k, m be two positive integers. We define the following measurable set,

$$B_k(m) = \left\{x \in X;\ \forall n \geq k,\ |\varphi_n(x) - \varphi(x)| < \frac{1}{m}\right\}. \tag{9}$$

A. Prove that $B_k(m+1) \subset B_k(m) \subset B_{k+1}(m)$.

B. Prove that for all positive integers m,

$$X = \bigcup_{k=1}^{\infty} B_k(m). \tag{10}$$

C. Prove that for all positive integers m,

$$\mu(X) = \lim_{k\to\infty} \mu(B_k(m)). \tag{11}$$

Hint: express $\bigcup_{k=1}^{\infty} B_k(m)$ as a countable union of measurable *mutually disjoint* sets, and observe that the resulting sum of measures is a telescopic series.

D. Let $\varepsilon > 0$. Prove that there exists a sequence of integers $\{k_m\}_{m\geq 1}$, such that

$$\mu(X) - \mu(B_{k_m}(m)) < \frac{\varepsilon}{2^m}. \tag{12}$$

E. Let $\varepsilon > 0$. We define

$$Y_\varepsilon = \bigcap_{m=1}^{\infty} B_{k_m}(m), \tag{13}$$

where the $\{k_m\}$ is the sequence defined in the previous question. Prove that Y_ε is measurable.

F. Prove that $\mu(X - Y_\varepsilon) < \varepsilon$, where the set difference $X - Y_\varepsilon$ is defined in (8).

G. Conclude that

$$\forall \varepsilon > 0,\ \exists Y_\varepsilon, \quad \text{such that } \mu(X - Y_\varepsilon) < \varepsilon, \text{ and } \varphi_n \text{ converges uniformly to } \varphi \text{ on } Y_\varepsilon. \tag{14}$$