APPM 5450 Spring 2015 Applied Analysis 2

Short Study Guide

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Intuitions and tricks

Utility and "under-appreciated" theorems

You may recall that some homework assignments made extensive use of these.

1. Prop. 5.30 regarding the open-mapping theorem (Thm. 5.23) and its converse

2. Corollary 6.15 and its implications, e.g., Theorem 8.17 and the unnumbered equation before Theorem 8.18, e.g., if $A \in \mathcal{B}(\mathcal{H})$,

$$\mathcal{H} = \overline{\operatorname{ran}(A)} \oplus \ker(A^*) \quad \text{and} \quad \mathcal{H} = \overline{\operatorname{ran}(A^*)} \oplus \ker(A)$$

3. Thm. 8.40: weak convergence iff norm is bounded and converges on a dense subset of the dual space.

Tricks/intuition

Showing an operator is **compact**: use the definition (maps bounded sets to precompact sets); or maps weakly-convergent sequences to strongly convergent (and for counter-example for this, use orthonormal sequences), e.g., Thm. 9.24; or is finite-rank or the limit of finite-rank operators. Theorem 9.16 and §9.4 give specific criterion for operators over Banach space, mainly related to being the limit of finite-rank operators. See also Prop. 5.43.

Another good trick for compactness/spectra is viewing the operator in the Fourier domain and converting multiplication/shifts to shifts/multiplication.

Counter-examples: see example 8.43 and example 12.61 for *oscillation*, *concentration* and *escape to infinity*. Also, use the left and right shift operators for a lot of examples. In general, try orthonormal sequences (e.g., in ℓ^2) or sequences of non-overlapping indicator functions (e.g., in L^p).

Basic facts for L spaces: L^p and ℓ^p are Banach for $1 \le p \le \infty$, Hilbert for $p = 2$, and separable for $1 \le p < \infty$. Theorem 12.51 shows that for $1 \le p < \infty$, C_c^∞ is dense in L^p. The $L^p(I)$ spaces are nested if I is bounded. Know the dual spaces of L^p and ℓ^p for the cases $p = 1$, $1 < p < \infty$ and $p = \infty$. Understand the ideas of equivalence classes, ess sup, and sets of zero measure.

Note that $f \in L^p$ iff $f^p \in L^1$.

Fourier transforms: A lot can be inferred just by knowing a few key transform pairs. First, note that the inverse transform $\mathcal{F}^{-1}$ is basically the same as the forward transform, since it is just a reflection, and hence we speak of Fourier transform "pairs" without paying attention to which is "f" and which is $\widehat{f}$. Some good pairs:

- f is a Gaussian, then $\widehat{f}$ is Gaussian too, with the opposite type of scaling (e.g., f wide means $\widehat{f}$ skinny, and vice-versa).

- f is the indicator function of $[-1, 1]$, $\widehat{f}$ is a sinc. Using Riemann-Lebesgue, this proves sinc $\notin L^1$, and also proves Riemann-Lebesgue does not hold on L^2.

- $f = \delta$ then $\widehat{f}$ is a constant function
- In general, f smooth implies $\widehat{f}$ decays quickly, and vice-versa. Note the relation between f' and $(ik)\widehat{f}(k)$

The Fourier transform on L^1 or on $\mathcal{S}$ is defined using the standard integral. In $\mathcal{S}^*$, it is defined weakly (i.e., in a distributional sense); on L^2, it is defined using the BLT theorem. We have Riemann-Lebesgue for $\mathcal{F}$ on L^1 but not on L^2. The Fourier transform maps $\mathcal{S}$ to $\mathcal{S}$, and $\mathcal{S}^*$ to $\mathcal{S}^*$, and L^2 to L^2. It does not map L^1 to L^1, so the inverse Fourier transform on the range of $\mathcal{F}(L^1)$ is defined in a distributional sense.

Fourier series: we use density arguments a lot, e.g., continuous functions are dense in $L(\mathbb{T})$, and trigonometric polynomials are dense in $C(\mathbb{T})$. We use convolutions to smooth functions, and convolutions with approximate identities will converge to the original function. Use Bessel's inequality a lot.

Fundamental things you should know by heart

Theorems

1. Bessel's inequality for orthonormal sets (Thm. 6.24), Parseval's equality for orthonormal bases (Thm. 6.26 part(c)), and the generalization of Parseval's (Thm. 6.28 Plancherel's (Thm. 11.37)
2. Riesz representation theorem
3. Banach-Alouglu (book has three versions: Thm. 5.61 (most general), Thm. 8.45 for Hilbert spaces, Thm. 12.62 for L^p for $1 < p < \infty$.
4. Spectral theorem (9.16) (after the proof, see paragraph discussing extensions: it also holds for normal operators, not just self-adjoint ones), and corollary 9.14 ($\sigma_r(A) = \emptyset$ if $A = A^*$).
5. Hölder and Minkowski
6. Sobolev embedding theorem (Thm. 7.9, and see subsequent paragraph for $\mathbb{R}^d$ with $d > 1$).
7. Riemann-Lebesgue
8. Monotone and Dominated Convergence Theorem; Fatou's lemma; Fubini's theorem.

Definitions

1. Concepts from last semester: (uniform) continuity, compactness, completeness, Banach/Hilbert spaces, dual spaces, Hahn-Banach theorem, strong/weak convergence, uniform vs strong-operator vs weak-operator convergence, weak-* convergence
2. Definitions like adjoint, self-adjoint, unitary/orthogonal, positive, coercive, resolvent, spectra, Schwartz space
3. Sobolev spaces (Primarily Def. 7.6 and Def. 11.38, but see also Def. 12.66)
4. Understand the basics of a σ-algebra, Borel σ-algebra, measurable sets, complete measure space, Lebesgue measure, measurable function, Lebesgue integral. Think of analogies with topology and continuous functions.
5. Know the concepts of weak-derivatives and distributional-derivatives, and how we define the Fourier transform of a distribution.