Intuitions and tricks

Utility and “under-appreciated” theorems

You may recall that some homework assignments made extensive use of these.

1. Prop. 5.30 regarding the open-mapping theorem (Thm. 5.23) and its converse

2. Corollary 6.15 and its implications, e.g., Theorem 8.17 and the unnumbered equation before Theorem 8.18, e.g., if \( A \in B(\mathcal{H}) \),

\[ \mathcal{H} = \text{ran}(A) \oplus \ker(A^*) \quad \text{and} \quad \mathcal{H} = \overline{\text{ran}(A^*)} \oplus \ker(A) \]

3. Thm. 8.40: weak convergence iff norm is bounded and converges on a dense subset of the dual space.

Tricks/intuition

Showing an operator is compact: use the definition (maps bounded sets to precompact sets); or maps weakly-convergent sequences to strongly convergent (and for counter-example for this, use orthonormal sequences), e.g., Thm. 9.24; or is finite-rank or the limit of finite-rank operators. Theorem 9.16 and §9.4 give specific criterion for operators over Banach space, mainly related to being the limit of finite-rank operators. See also Prop. 5.43.

Another good trick for compactness/spectra is viewing the operator in the Fourier domain and converting multiplication-shifts to shifts/multiplication.

Counter-examples: see example 8.43 and example 12.61 for oscillation, concentration and escape to infinity. Also, use the left and right shift operators for a lot of examples. In general, try orthonormal sequences (e.g., in \( \ell^2 \)) or sequences of non-overlapping indicator functions (e.g., in \( L^p \)).

Basic facts for \( L \) spaces: \( L^p \) and \( \ell^p \) are Banach for \( 1 \leq p \leq \infty \), Hilbert for \( p = 2 \), and separable for \( 1 \leq p < \infty \). Theorem 12.51 shows that for \( 1 \leq p < \infty \), \( C_c^\infty \) is dense in \( L^p \). The \( L^p(I) \) spaces are nested if \( I \) is bounded. Know the dual spaces of \( L^p \) and \( \ell^p \) for the cases \( p = 1, 1 < p < \infty \) and \( p = \infty \). Understand the ideas of equivalence classes, ess sup, and sets of zero measure.

Fourier transforms: A lot can be inferred just by knowing a few key transform pairs. First, note that the inverse transform \( F^{-1} \) is basically the same as the forward transform, since it is just a reflection, and hence we speak of Fourier transform “pairs” without paying attention to which is “\( f \)” and which is \( \hat{f} \). Some good pairs:

- \( f \) is a Gaussian, then \( \hat{f} \) is Gaussian too, with the opposite type of scaling (e.g., \( f \) wide means \( \hat{f} \) skinny, and vice-versa).

- \( f \) is the indicator function of \([-1, 1]\), \( \hat{f} \) is a sinc. Using Riemann-Lebesgue, this proves sinc \( \not\in L^1 \), and also proves Riemann-Lebesgue does not hold on \( L^2 \).
• \( f = \delta \) then \( \hat{f} \) is a constant function

• In general, \( f \) smooth implies \( \hat{f} \) decays quickly, and vice-versa. Note the relation between \( f' \) and \((ik)f(k)\)

The Fourier transform on \( L^1 \) or on \( S \) is defined using the standard integral. In \( S^* \), it is defined weakly (i.e., in a distributional sense); on \( L^2 \), it is defined using the BLT theorem. We have Riemann-Lebesgue for \( \mathcal{F} \) on \( L^1 \) but not on \( L^2 \). The Fourier transform maps \( S \) to \( S \), and \( S^* \) to \( S^* \), and \( L^2 \) to \( L^2 \). It does not map \( L^1 \) to \( L^1 \), so the inverse Fourier transform on the range of \( \mathcal{F}(L^1) \) is defined in a distributional sense.

**Fourier series:** we use density arguments a lot, e.g., continuous functions are dense in \( L^p(T) \), and trigonometric polynomials are dense in \( C(T) \). We use convolutions to smooth functions, and convolutions with approximate identities will converge to the original function. Use Bessel’s inequality a lot.

**Fundamental things you should know by heart**

**Theorems**
1. Bessel’s inequality for orthonormal sets (Thm. 6.24), Parseval’s equality for orthonormal bases (Thm. 6.26 part(c)), and the generalization of Parseval’s (Thm. 6.28 Plancherel’s (Thm. 11.37)
2. Riesz representation theorem
3. Banach-Alouglu (book has three versions: Thm. 5.61 (most general), Thm. 8.45 for Hilbert spaces, Thm. 12.62 for \( L^p \) for \( 1 < p < \infty \).
4. Spectral theorem (9.16) (after the proof, see paragraph discussing extensions: it also holds for normal operators, not just self-adjoint ones), and corollary 9.14 (\( \sigma_r(A) = \emptyset \) if \( A = A^* \)).
5. Hölder and Minkowski
6. Sobolev embedding theorem (Thm. 7.9, and see subsequent paragraph for \( \mathbb{R}^d \) with \( d > 1 \)).
7. Riemann-Lebesgue
8. Monotone and Dominated Convergence Theorem; Fatou’s lemma; Fubini’s theorem.

**Definitions**
1. Concepts from last semester: (uniform) continuity, compactness, completeness, Banach/Hilbert spaces, dual spaces, Hahn-Banach theorem, strong/weak convergence, uniform vs strong-operator vs weak-operator convergence, weak-* convergence
2. Definitions like adjoint, self-adjoint, unitary/orthogonal, positive, coercive, resolvent, spectra, Schwartz space
3. Sobolev spaces (Primarily Def. 7.6 and Def. 11.38, but see also Def. 12.66)
4. Understand the basics of a \( \sigma \)-algebra, Borel \( \sigma \)-algebra, measurable sets, complete measure space, Lebesgue measure, measurable function, Lebesgue integral. Think of analogies with topology and continuous functions.
5. Know the concepts of weak-derivatives and distributional-derivatives, and how we define the Fourier transform of a distribution.