APPM 5450 Spring 2015 Applied Analysis 2

Study Guide or:

How I learned to Stop Worrying and Love Analysis. Version 5*

Spring 2015, updated May 11, 2016

Chapter 7: Fourier Series

7.1 The Fourier Basis

The Torus ($\mathbb{T}$): A 2π-periodic function on $\mathbb{R}$ may be identified with a function on a circle, or one-dimensional torus, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. which we define by identifying points in $\mathbf{T}$ that differ by $2\pi n$ for some $n \in \mathbb{Z}$

Inner Product ($L^2(\mathbb{T})$):

$$\langle f, g\rangle = \int_{\mathbb{T}} \overline{f(x)} g(x) dx$$

Fourier Basis: The Fourier basis elements are the functions

$$e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}.$$

Fourier Series: Any function $f \in L^2(\mathbb{T})$ may be expanded in a Fourier series as

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx},$$

where the equality means convergence of the partial sums of f in the L^2-norm, and

$$\hat{f}_n = \langle e_n, f\rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x) e^{-inx} dx.$$

Definition 7.1: A family of functions $\{\varphi_n \in C(\mathbb{T}) | n \in \mathbb{N}\}$ is an *approximate identity* if:

(a) $\varphi_n(x) \geq 0$;

(b) $\int_{\mathbb{T}} \varphi_n(x) dx = 1$ for every $n \in \mathbb{N}$;

(c) $\lim_{n\to\infty} \int_{\delta \leq |x| \leq \pi} \varphi_n(x) dx = 0$ for every $\delta > 0$.

*Changes from version 1: In Def 7.6, fixed H^1 to read H^k at the appropriate spot; also, typo corrections; Changes from version 2: more H^k typo fixes in Lemma 7.8 and Thm 7.9; Changes from version 3: typo fixes; Changes from v4 to v5: typos fixed 2016 (extra absolute value in Thm 12.41 part (b), and strict inequality in Def'n 11.38)

Theorem 7.2: If $\{\varphi_n \in C(\mathbb{T}) | n \in \mathbb{N}\}$ is and approximate identity and $f \in C(\mathbb{T})$, then $\varphi_n * f$ converges uniformly to f as $n \to \infty$.

Theorem 7.3: The trigonometric polynomials are dense in $C(\mathbb{T})$ with respect to the uniform norm.

Proposition 7.4: If $f, g \in L^2(\mathbb{T})$, then $f * g \in C(\mathbb{T})$ and

$$\|f * g\|_\infty \leq \|f\|_2 \|g\|_2.$$

Theorem 7.5 (Convolution): If $f, g \in L^2(\mathbb{T})$, then

$$\widehat{(f * g)}_n = \sqrt{2\pi} \hat{f}_n \hat{g}_n.$$

7.2 Fourier Series of Differentiable Functions

Definition 7.6: The *Sobolev space* $H^1(\mathbb{T})$ consists of all functions

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx} \in L^2(\mathbb{T})$$

such that

$$\sum_{n=-\infty}^{\infty} n^2 \left|\hat{f}_n\right|^2 < \infty.$$

The weak L^2-derivative $f' \in L^2(\mathbb{T})$ of $f \in H^1(\mathbb{T})$ is defined by the L^2-convergent Fourier series

$$f'(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} in\hat{f}_n e^{inx} \in L^2(\mathbb{T}).$$

More generally

$$H^k(\mathbb{T}) = \left\{ f \in L^2(\mathbb{T}) \,\middle|\, f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \sum_{n=-\infty}^{\infty} |n|^{2k} |c_n|^2 < \infty \right\},$$

for $k \geq 0$.

Definition 7.7: A function $f \in L^2(\mathbb{T})$ belongs to $H^1(\mathbb{T})$ if there is a constant M such that

$$\left| \int_{\mathbb{T}} f \varphi' dx \right| \leq M \|\varphi\|_{L^2} \quad \text{for all } \varphi \in C^1(\mathbb{T})$$

for $f \in H^1(\mathbb{T})$, then the *weak derivative* f' of f is the unique element of $L^2(\mathbb{T})$ such that

$$\int_{\mathbb{T}} f' \varphi dx = -\int_{\mathbb{T}} f \varphi' dx \quad \text{for all } \varphi \in C^1(\mathbb{T}).$$

Lemma 7.8: Suppose that $f \in H^k(\mathbb{T})$ for $k > 1/2$. Let

$$S_N(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} \hat{f}_n e^{inx}$$

be the Nth partial sum of the Fourier series of f, and define

$$\left\| f^{(k)} \right\| = \left(\sum_{n=-\infty}^{\infty} |n|^{2k} |\hat{f}_n|^2 \right)^{\frac{1}{2}}.$$

Then there is a constant C_k, independent of f, such that

$$\|S_N - f\|_\infty \leq \frac{C_k}{N^{k-\frac{1}{2}}} \left\| f^{(k)} \right\|,$$

and (S_N) converges uniformly to f as $N \to \infty$.

Theorem 7.9: If $f \in H^k(\mathbb{T})$ for $k > 1/2$, then $f \in C(\mathbb{T})$. More generally: If $f \in H^k(\mathbb{T}^d)$ and $k > j + d/2$ then $f \in C^j(\mathbb{T}^d)$.

Chapter 8: Bounded Linear Operators on a Hilbert Space

8.1 Orthogonal Projections

Definition 8.1: A *projection* on a linear space X is a linear map $P : X \to X$ such that

$$P^2 = P.$$

Theorem 8.2: Let X be a linear space.

(a) If $P : X \to X$ is a projection, then $X = \text{ran}P \oplus \text{ker}P$.

(b) If $X = M \oplus N$, where M and N are linear subspaces of X, then there is a projection $P : X \to X$ with $\text{ran}P = M$ and $\text{ker}P = N$.

Definition 8.3: An *orthogonal projection* on a Hilbert space $\mathcal{H}$ is a linear map $P : \mathcal{H} \to \mathcal{H}$ that satisfies

$$P^2 = P, \quad \langle Px, y\rangle = \langle x, Py\rangle \quad \text{for all } x, y \in \mathcal{H}.$$

An orthogonal projection is necessarily bounded.

Proposition 8.4: If P is a nonzero orthogonal projection, then $\|P\| = 1$.

Theorem 8.5: Let $\mathcal{H}$ be a Hilbert space.

(a) If P is an orthognal projection on $\mathcal{H}$, then $\text{ran}P$ is closed, and

$$\mathcal{H} = \text{ran}P \oplus \text{ker}P$$

is the orthogonal direct sum of $\text{ran}P$ and $\text{ker}P$.

(b) If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, then there is an orthogonal projection P on $\mathcal{H}$ with $\text{ran}P = \mathcal{M}$ and $\text{ker}P = \mathcal{M}^{\perp}$.

Even and Odd Projections: The space $L^2(\mathbb{R})$ is the orthogonal direct sum of the space $\mathcal{M}$ of even functions and the space $\mathcal{N}$ of the odd functions. The orthogonal projections P and Q of $\mathcal{H}$ onto $\mathcal{M}$ and $\mathcal{N}$, respectively, are given by

$$Pf(x) = \frac{f(x) + f(-x)}{2}, \qquad Qf(x) = \frac{f(x) - f(-x)}{2}.$$

note that $I - P = Q$.

8.2 The Dual of a Hilbert Space

Linear Functional: A *linear functional* on a complex Hilbert space $\mathcal{H}$ is a linear mapy from $\mathcal{H}$ to $\mathbb{C}$. A linear function φ is bounded, or continuous, if there exist a constant M such that

$$|\varphi(x)| \le M\|x\| \quad \text{for all } x \in \mathcal{H}.$$

The norm of a bounded linear function φ is

$$\|\varphi\| = \sup_{\|x\|=1} |\varphi(x)|.$$

if $y \in \mathcal{H}$, then

$$\varphi_y(x) = \langle y, x \rangle$$

is a bounded linear functional on $\mathcal{H}$, with $\|\varphi_y\| = \|y\|$.

Theorem 8.12 (Riesz representation): If φ is a bounded linear functional on a Hilbert space $\mathcal{H}$, then there is a unique vector $y \in \mathcal{H}$ such that

$$\varphi(x) = \langle y, x \rangle \quad \text{for all } x, y \in \mathcal{H}.$$

8.3 The Adjoint of an Operator

Adjoint Operator: Let $A \in \mathcal{B}(\mathcal{H})$ then there exist an unique $A^* \in \mathcal{B}(\mathcal{H})$ (known as the *adjoint*) such that

$$\langle x, Ay \rangle = \langle A^* x, y \rangle \quad \text{for all } x, y \in \mathcal{H}.$$

Left and Right Shift Operators: Suppose that S and T are the right and left shift operators on the sequence space $\ell^2(\mathbb{N})$, defined by

$$S(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots), \quad T(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Then $T = S^*$.

Theorem 8.17: If $A : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator, then

$$\overline{\text{ran} A} = (\text{ker} A^*)^{\perp}, \quad \text{ker} A = (\text{ran} A^*)^{\perp}.$$

Theorem 8.18: Suppose that $A : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on a Hilbert space $\mathcal{H}$ with closed range. Then the equation $Ax = y$ has a solution for x if and only if y is orthogonal to $\text{ker} A^*$.

8.4 Self-Adjoint and Unitary Operators

Definition 8.23: A bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is *self-adjoint* if $A^* = A$. Equivalently, abounded linear operator A on $\mathcal{H}$ is self-adjoint if and only if

$$\langle x, Ay \rangle = \langle Ax, y \rangle \quad \text{for all } x, y \in \mathcal{H}.$$

Nonnegative, Positive/Positive Definite, Coercive: Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then A is

- *nonnegative* if $\langle x, Ax \rangle \geq 0$ for all $x \in \mathcal{H}$,
- *positive* or *positive definite* if $\langle x, Ax \rangle > 0$ for all nonzero $x \in \mathcal{H}$,
- *coercive* if there exists a $c > 0$ such that $\langle x, Ax \rangle \geq c\|x\|^2$ for all $x \in \mathcal{H}$.

Lemma 8.26: If A is a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$, then

$$\|A\| = \sup_{\|x\|=1} |\langle x, Ax\rangle|.$$

Corollary 8.27: If A is a bounded self-adjoint operator on a Hilbert space then $\|A^*A\| = \|A\|^2$. If A is self-adjoint, then $\|A^2\| = \|A\|^2$.

Definition 8.28: A linear map $U : \mathcal{H}_1 \to \mathcal{H}_2$ between real or complex Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ it is said to be *orthogonal* or *unitary*, respectively, if it is invertible and if

$$\langle Ux, Uy\rangle_{\mathcal{H}_2} = \langle x, y\rangle_{\mathcal{H}_1} \quad \text{for all } x, y \in \mathcal{H}_1.$$

Two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are *isomorphic* as Hilbert spaces if there is a unitary linear map between them. If $U : \mathcal{H} \to \mathcal{H}$ (i.e $\mathcal{H}_1 = \mathcal{H}_2$), then U is unitary if and only if $U^*U = UU^* = I$.

Example 8.30: If A is a bounded self-adjoint operator, then

$$e^{iA} = \sum_{n=0}^{\infty} \frac{1}{n!}(iA)^n$$

is unitary.

8.6 Weak Convergence in a Hilbert Space

Weak Convergence: A sequence (x_n) in a Hilbert space $\mathcal{H}$ converges *weakly* to $x \in \mathcal{H}$ if

$$\lim_{n\to\infty} \langle x_n, y\rangle = \langle x, y\rangle \quad \text{for all } y \in \mathcal{H}.$$

Theorem 8.39 (Uniform Boundedness): Suppose that

$$\{\varphi_n : X \to \mathbb{C} | n \in \mathbb{N}\}$$

is a set of linear functionals on a Banach Space X such that the set of complex numbers $\{\varphi_n(x) | n \in \mathbb{N}\}$ is bounded for each $x \in X$. Then $\{\|\varphi_n\| \; | n \in \mathbb{N}\}$ is bounded.

Theorem 8.40: Suppose that (x_n) is a sequence in a Hilbert space $\mathcal{H}$ and D is a dense subset of $\mathcal{H}$. Then (x_n) converges weakly to x if and only if:

(a) $\|x_n\| \leq M$ for some constant M;

(b) $\langle x_n, y\rangle \to \langle x, y\rangle$ as $n \to \infty$ for all $y \in D$.

Note: this is **extremely useful** for showing weak convergence by letting D be a basis for $\mathcal{H}$ as in Example 8.41.

Proposition 8.44: If (x_n) converges weakly to x, then

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

If, in addition,

$$\lim_{n\to\infty} \|x_n\| = \|x\|,$$

then (x_n) converges strongly to x.

Theorem 8.45 (Banach-Alaoglu): The closed unit ball of a Hilbert space is weakly compact.

Definition 8.47: Let $f : C \to \mathbb{R}$ be a real-valued function on a convex subset C of a real or complex linear space. Then f is *convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in C$ and $0 \leq t \leq 1$. If we have a strict inequality in this equation whenever $x \neq y$ and $0 < t < 1$, then f is *strictly convex*.

Theorem 8.48 (Mazur): If (x_n) converges weakly to x in a Hilbert space, then there is a sequence (y_n) of finite convex combinations of $\{x_n\}$ such that (y_n) converges strongly to x.

Chapter 9: Spectrum of Bounded Linear Operators

9.2 The Spectrum

A Quick Note: A great reference for the spectrum of common operators (such as general multiplication operators and the shift operators) check out this wonderful pdf. http://www.math.ubc.ca/~feldman/m511/spectralExamples.pdf

Definition 9.3: The *resolvent set* of on operator $A \in \mathcal{B}(\mathcal{H})$, denoted by $\rho(A)$, is the set of complex numbers λ such that $(A - \lambda I) : \mathcal{H} \to \mathcal{H}$ is one-to-one and onto. The *spectrum* of A, denoted by $\sigma(A)$, is the complement of the resolvent set in $\mathbb{C}$, meaning that $\sigma(A) = \mathbb{C} \backslash \rho(A)$

Definition 9.4: Suppose that A is a bounded linear operator on a Hilbert space $\mathcal{H}$.

(a) The *point spectrum* of A consists of all $\lambda \in \sigma A$ such that $A - \lambda I$ is not one-to-one. In this case λ is called an *eigenvalue* of A.

(b) The *continuous spectrum* of A consists of all $\lambda \in \sigma A$ such that $A - \lambda I$ is one-to-one but not onto, and $\operatorname{ran}(A - \lambda I)$ is dense in $\mathcal{H}$.

(c) The *residual spectrum* of A consists of all $\lambda \in \sigma A$ such that $A - \lambda I$ is one-to-one but not onto, and $\operatorname{ran}(A - \lambda I)$ is not dense in $\mathcal{H}$.

Proposition 9.6: If A is a bounded linear operator on a Hilbert space, then the resolvent set $\rho(A)$ is an open subset of $\mathbb{C}$ that contains the exterior disc $\{\lambda \in \mathbb{C} \mid |\lambda| > \|A\|\}$. The resolvent R_λ is an operator-valued analytic function of λ defined on $\rho(A)$.

Proposition 9.7: Let the *spectral radius* be defined as

$$r(A) = \sup\{|\lambda| \mid \lambda \in \sigma(A)\}.$$

If A is a bounded linear operator, then

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

If A is self-adjoint, then $r(A) = \|A\|$.

Proposition 9.8: The spectrum of a bounded operator on a Hilbert space is nonempty

The Spectral Theorem for Compact, Self-Adjoint Operators

Lemma 9.9: The eigenvalues of a bounded, self-adjoint operator are real, and the eigenvectors associated with different eigenvalues are orthogonal.

Invariant Subspace: A linear subspace $\mathcal{M}$ of $\mathcal{H}$ is called an *invariant subspace* of a linear operator A on $\mathcal{H}$ if $Ax \in \mathcal{M}$ for all $x \in \mathcal{M}$

Lemma 9.11: If A is a bounded, self-adjoint operator on a Hilbert space H and $\mathcal{M}$ is an invariant subspace of A, then $\mathcal{M}^\perp$ is an invariant subspace of A.

Proposition 9.12: If λ belongs to the residual spectrum of a bounded operator A on a Hilbert space, then $\overline{\lambda}$ is an eigenvalue of A^*.

Lemma 9.13 If A is a bounded, self-adjoint operator on a Hilbert space, then the spectrum of A is real and is contained in the interval $[-\|A\|, \|A\|]$.

Corollary 9.14: The residual spectrum of a bounded, self-adjoint operator is empty.

Proposition 9.15: A nonzero eigenvalue of a compact, self-adjoint operator has finite multiplicity. A countably infinite set of nonzero eigenvalues has zero as an accumulation point, and no other accumulation points.

Theorem 9.16 (Spectral Theorem for Compact, Self-Adjoint Operators): Let $A : \mathcal{H} \to \mathcal{H}$ be a compact, self-adjoint operator on a Hilbert space $\mathcal{H}$. There is an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of A. The nonzero eigenvalue of A form a finite or countably infinite set $\{\lambda_k\}$ of real numbers, and

$$A = \sum_k \lambda_k P_k,$$

where P_k is the orthogonal projection onto the finite-dimensional eigenspace of eigenvectors with eigenvalues λ_k. If the number of nonzero eigenvalues is countably infinite then the series above converges to A in operator norm.

9.4 Compact Operators

Theorem 9.17: Let E be a subset of an infinite-dimensional, separable Hilbert space $\mathcal{H}$.

(a) If E is precompact, then for every orthonormal set $\{e_n \mid n \in \mathbb{N}\}$ and every $\epsilon > 0$, there is an N such that

$$\sum_{n=N+1}^{\infty} |\langle e_n, x\rangle|^2 < \epsilon \quad \text{for all } x \in E.$$

(b) if E is bounded and there is an orthonormal basis $\{e_n\}$ of $\mathcal{H}$ with the property that for every $\epsilon > 0$ there is an N such that the sum in (a) holds, then E is precompact.

Example 9.19: The diagonal operator $A : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by

$$A(x_1, x_2, x_3, \dots, x_n, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \dots, \lambda_n x_n, \dots),$$

where $\lambda_n \in \mathbb{C}$ is compact if and only if $\lambda_n \to 0$ as $n \to \infty$. Any compact, normal operator on a separable Hilbert space is unitarily equivalent to such a diagonal operator. Note: this implies that the identity operator on a Hilbert space $\mathcal{H}$ is compact if and only if $\mathcal{H}$ is finite-dimensional.

Definition 9.20: A bounded linear operator A on a separable Hilbert space $\mathcal{H}$ is *Hilbert-Schmidt* if there is an orthonormal basis $\{e_n \mid n \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty.$$

If A is a Hilbert-Schmidt operator, then

$$\|A\|_{HS} = \sqrt{\sum_{n=1}^{\infty} \|Ae_n\|^2}$$

is called the *Hilbert-Schmidt norm* of A.

Theorem 9.21: A Hilbert-Schmidt operator is compact.

Theorem 9.24: A bounded linear operator on a Hilbert space is compact if and only if it maps weakly convergent sequences into strongly convergent sequences.

9.5 Functions of Operators

Theorem 9.25 (Spectral Mapping): If A is a compact, self-adjoint operator on a Hilbert space and $f : \sigma(A) \to \mathbb{C}$ is continuous, then

$$\sigma(f(A)) = f(\sigma(A)).$$

Here $\sigma(f(A))$ is the spectrum of $f(A)$, and $f(\sigma(A))$ is the image of the spectrum of A under f,

$$f(\sigma(A)) = \{\mu \in \mathbb{C} \mid \mu = f(\mu) \text{ for some } \lambda \in \sigma(A)\}.$$

Chapter 11: Distributions and the Fourier Transform

11.1 The Schwartz Space

Multi-Index: A *multi-index*

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n \quad \text{with } \mathbb{Z}_+ = \{n \in \mathbb{Z} \mid n \geq 0\}$$

is an n-tuple of nonnegative integers $a_i \geq 0$. For multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$ and $x \in \mathbb{R}^n$, we define

(a) $|\alpha| = \sum_{i=1}^n \alpha_i$,

(b) $\alpha! = \prod_{i=1}^n \alpha_i!$,

(c) $\alpha + \beta = (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$,

(d) $\alpha \geq \beta$ if and only if $\alpha_i \geq \beta_i$ for $i = 1, \dots, n$,

(e) $\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$,

(f) $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$,

(g) $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.

Leibnitz Rule: The Leibnitz rule for the derivative of the product of $f, g \in C^k(\mathbb{R}^n)$ may be written as

$$\partial^\alpha (fg) = \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta!\gamma!} (\partial^\beta f)(\partial^\gamma g).$$

Definition 11.1: The *Schwartz space* $\mathcal{S}(\mathbb{R}^n)$, or $\mathcal{S}$ for short, consists of all functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that

$$p_{\alpha,\beta}(\varphi) = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)|$$

is finite for every pair of multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$. If $\varphi \in \mathcal{S}$, then for every $d \in \mathbb{N}$ and $\alpha \in \mathbb{Z}_+^n$ there is a constant $C_{d,\alpha}$ such that

$$|\partial^\alpha \varphi(x)| \leq \frac{C_{d,\alpha}}{(1 + |x|^2)^d/2} \quad \text{for all } x \in \mathbb{R}^n.$$

Thus an element of $\mathcal{S}$ is a smooth function such that the function and all of its derivatives decay faster than any polynomial as $|x| \to \infty$. Elements of $\mathcal{S}$ are called *Schwartz functions*, or *test functions*.

Definition 11.2: Suppose that X is a real or complex linear space. A function $p : X \to \mathbb{R}$ is a *seminorm* on X if it has the following properties:

(a) $p(x) \geq 0$ for all $x \in X$;

(b) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$;

(c) $p(\lambda x) = |\lambda| p(x)$ for every $x \in X$ and $\lambda \in \mathbb{C}$.

A seminorm has the same properties as a norm, except that $p(x) = 0$ does not need to imply $x = 0$. If a family of seminorms $\{p_1, \dots, p_n\}$ is finite and seperates points, then

$$\|x\| = p_1(x) + \ldots + p_n(x)$$

defines a norm on X

Proposition 11.3: The Schwartz space $\mathcal{S}$ with the metrizable topology generated by the countable family of seminorms

$$\{p_{\alpha,\beta} \mid \alpha, \beta \in \mathbb{Z}_+^n\},$$

where $p_{\alpha,\beta}$ is given by

$$\|\varphi\|_{\alpha,\beta} = p_{\alpha,\beta}(\varphi) = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)|,$$

is complete.

Proposition 11.4: For each $\alpha \in \mathbb{Z}_+^n$, the partial differentiation operator $\partial^\alpha : \mathcal{S} \to \mathcal{S}$ is continuous linear operator on $\mathcal{S}$.

11.2 Tempered Distributions

Tempered Distributions: The topological dual space of $\mathcal{S}$, denoted by $\mathcal{S}^*$ or $\mathcal{S}'$, is the space of continuous linear functionals $T : \mathcal{S} \to \mathbb{C}$. Elements of $\mathcal{S}^*$ are called *tempered distributions*. Since $\mathcal{S}$ is a metric space, a functional $T : \mathcal{S} \to \mathbb{C}$ is continuous if and only if for every convergent sequence $\varphi_n \to \varphi$ in $\mathcal{S}$, we have

$$\lim_{n\to\infty} T(\varphi_n) = T(\varphi).$$

The continuity of a linear functional T is implied by an estimate of the form

$$|T(\varphi)| \leq \sum_{|\alpha|,|\beta|\leq d} c_{\alpha,\beta} \|\varphi\|_{\alpha,\beta}$$

for some $d \in \mathbb{Z}_+$ and constants $c_{\alpha,\beta} \geq 0$.

Example 11.5: The fundamental example of a distribution is the *delta function*. We define $\delta : \mathcal{S} \to \mathbb{C}$ by evaluation at 0:

$$\delta(\varphi) = \varphi(0).$$

11.3 Operations on Distributions

Example 11.9: If $T = \delta$ is the delta function, then

$$\langle f\delta, \varphi\rangle = \langle \delta, f\varphi\rangle = f(0)\varphi(0) = \langle f(0)\delta, \varphi\rangle.$$

Hence, $f\delta = f(0)\delta$.

Definition 11.10: Suppose that T is a tempered distribution and α is a multi-index. the αth *distributional derivative* of T is the tempered distribution $\partial^\alpha T$ defined by

$$\langle \partial^\alpha T, \varphi\rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi\rangle \quad \text{for all } \varphi \in \mathcal{S}.$$

Theorem 11.11: For every $T \in \mathcal{S}^*$ there is a continuous function $f : \mathbb{R}^n \to \mathbb{C}$ of polynomial growth and a multi-index $\alpha \in \mathbb{Z}_+^n$ such that $T = \partial^\alpha f$.

Example 11.14: The derivative of the one-dimensional delta function δ is given by

$$\langle \delta', \varphi\rangle = -\langle \delta, \varphi'\rangle = -\varphi'(0).$$

More generally, the kth distributional derivative of δ is given by

$$\langle \delta^{(k)}, \varphi\rangle = (-1)^k \varphi^{(k)}(0).$$

Example 11.16: For each $h \in \mathbb{R}^n$, we define the *translation operator* $\tau_h : \mathcal{S} \to \mathcal{S}$ by

$$\tau_h f(x) = f(x - h).$$

We therefore define the translation $\tau_h T$ of a distribution T by

$$\langle \tau_h T, \varphi \rangle = \langle T, \tau_{-h}\varphi \rangle \quad \text{for all } \varphi \in \mathcal{S}.$$

For instance, we have $\delta_{x_0} = \tau_{x_0}\delta$.

Example 11.17: Let $R : \mathcal{S} \to \mathcal{S}$ be the *reflection operator*,

$$Rf(x) = f(-x).$$

Thus, for $T \in \mathcal{S}^*$, we defined the reflection $RT \in \mathcal{S}^*$ by

$$\langle RT, \varphi \rangle = \langle T, R\varphi \rangle \quad \text{for all } \varphi \in \mathcal{S}.$$

Proposition 11.18: For any $\varphi, \psi, \omega \in \mathcal{S}$, we have:

(a) $\varphi * \psi = \psi * \varphi$,

(b) $(\varphi * \psi) * \omega = \varphi * (\psi * \omega)$,

(c) $\tau_h(\varphi * \psi) = (\tau_h\varphi) * \psi = \varphi * (\tau_h\psi)$ for every $h \in \mathbb{R}^n$.

11.4 The Convergence of Distributions

Convergence in Distribution Space: Let (T_n) be a sequence in $\mathcal{S}^*$. We say that (T_n) converges to $T \in \mathcal{S}^*$ if and only if

$$\lim_{n\to\infty} \langle T_n, \varphi \rangle = \langle T, \varphi \rangle \quad \text{for every } \varphi \in \mathcal{S}.$$

We denote convergence is the space of distributions by $T_n \rightharpoonup T$ as $n \to \infty$.

Proposition 11.22: For $n \in \mathbb{N}$, let

$$\sigma_n(x) = \frac{\sin(nx)}{\pi x}.$$

Then $\sigma_n \rightharpoonup \delta$ in $\mathcal{S}^*$ as $n \to \infty$.

Theorem 11.23: The Schwartz space is dense in the space of tempered distributions.

11.5 The Fourier Transform of Test Functions

Definition 11.24: If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then the *Fourier transform* $\hat{\varphi} : \mathbb{R}^n \to \mathbb{C}$ is the function defined by

$$\hat{\varphi}(k) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(x) e^{-ik\cdot x} dx \quad \text{for } k \in \mathbb{R}^n.$$

We define the Fourier transform operator $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ by $\mathcal{F}\varphi = \hat{\varphi}$.

Eigenvalues of The Fourier Transform: Knowing that $\mathcal{F}^2\varphi = R\varphi$ where R is a reflection. It is easy to see that $\mathcal{F}^4\varphi = \varphi$ leading to the characteristic equation $\lambda^4 = 1$. It turns out that the eigenvalues of $\mathcal{F}$ are

$$\sigma_p(\mathcal{F}) = \{\pm 1, \pm i\}.$$

Proposition 11.25: If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then:

(a) $\hat{\varphi} \in C^\infty(\mathbb{R}^n)$, and

$$\partial^\alpha \hat{\varphi} = \mathcal{F}[(-ix)^\alpha \varphi];$$

(b) $k^\alpha \hat{\varphi}$ is bounded for every multi-index $\alpha \in \mathbb{Z}_+^n$, and

$$(ik)^\alpha \hat{\varphi} = \mathcal{F}[\partial^\alpha \varphi].$$

The Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a continuous linear map on $\mathcal{S}(\mathbb{R}^n)$.

Proposition 11.27: If $\varphi, \psi \in \mathcal{S}$ and $h \in \mathbb{R}^n$, then:

$$\widehat{\tau_h \varphi} = e^{-ik\cdot h}\hat{\varphi},$$

$$\widehat{e^{ix\cdot h}\varphi} = \tau_h \hat{\varphi},$$

$$\widehat{\varphi * \psi} = (2\pi)^{n/2}\hat{\varphi}\hat{\psi}.$$

Definition 11.28: If $\varphi \in \mathcal{S}$, then the *inverse Fourier transform* $\check{\varphi}$ is given by

$$\check{\varphi}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ik\cdot x}\varphi(k)dk.$$

We define $\mathcal{F}^* : \mathcal{S} \to \mathcal{S}$ by $\mathcal{F}^*\varphi = \check{\varphi}$.

Proposition 11.29: The map $\mathcal{F}^*$ is a continuous linear transformation on $\mathcal{S}$, and $\mathcal{F}^* = \mathcal{F}^{-1}$.

11.6 The Fourier Transform of Tempered Distributions

Definition 11.30: The Fourier transform of a tempered distribution T is the tempered distribution $\hat{T} = \mathcal{F}T$ defined by

$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle \quad \text{for all } \varphi \in \mathcal{S}.$$

The inverse Fourier transform $\check{T} = \mathcal{F}^{-1}T$ on $\mathcal{S}^*$ is defined by

$$\langle \check{T}, \varphi \rangle = \langle T, \check{\varphi} \rangle \quad \text{for all } \varphi \in \mathcal{S}.$$

The map $\mathcal{F} : \mathcal{S}^* \to \mathcal{S}^*$ is a continuous, one-to-one transformation of $\mathcal{S}^*$ onto itself.

Example 11.31: The Fourier transform of the delta function is

$$\langle \hat{\delta}, \varphi \rangle = \langle \delta, \hat{\varphi} \rangle = \hat{\varphi}(0) = \frac{1}{(2\pi)^{n/2}} \int \varphi(x)dx = \frac{1}{(2\pi)^{n/2}} \langle 1, \varphi \rangle.$$

Hence, the Fourier transform of the delta function is

$$\hat{\delta} = \frac{1}{(2\pi)^{n/2}}.$$

11.7 The Fourier Transform of L^1

Convergence of the Fourier Integral: The Fourier integral (Fourier transform on L^1-functions)

$$\hat{f}(k) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ik\cdot x} dx$$

converges if and only if $f \in L^1(\mathbb{R}^n)$, meaning that

$$\int_{\mathbb{R}^n} |f(x)| dx < \infty.$$

Theorem 11.34 (Riemann-Lebesgue): If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$, and

$$(2\pi)^{n/2} \|\hat{f}\|_\infty \leq \|f\|_1.$$

Theorem 11.35 (Convolution): If $f, g \in L^1(\mathbb{R}^n)$, then $f * g \in L^1(\mathbb{R}^n)$ and

$$\widehat{f * g} = (2\pi)^{n/2} \hat{f} \hat{g}.$$

11.8 The Fourier Transform of L^2

Theorem 11.37 (Plancherel): The Fourier Transform $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a unitary map. For every $f, g \in L^2(\mathbb{R}^n)$, we have

$$(\hat{f}, \hat{g}) = (f, g),$$

where

$$(f, g) = \int_{\mathbb{R}^n} \overline{f(x)} g(x) dx$$

and $\hat{f} = \mathcal{F} f$. In particular,

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \int_{\mathbb{R}^n} |\hat{f}(k)|^2 dk.$$

Definition 11.38: Let $s \in \mathbb{R}$. The Sobolev space $H^s(\mathbb{R}^n)$ consists of all distributions $f \in \mathcal{S}^*$ whose Fourier transform $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ is a regular distribution and

$$\int_{\mathbb{R}^n} (1 + |k|^2)^s |\hat{f}(k)|^2 dk < \infty.$$

Chapter 12: Measure Theory and Function Spaces

12.1 Measures

Definition 12.1: A *σ-algebra* on a set X is a collection of $\mathcal{A}$ of subsets of X such that:

(a) $\emptyset \in \mathcal{A}$;

(b) If $A \in \mathcal{A}$, then $A^c = X \backslash A \in \mathcal{A}$;

(c) If $\{A_i \mid i \in \mathbb{N}\}$ is a countable family of sets in $\mathcal{A}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

A *measurable space* $(X, \mathcal{A})$ is a set X and a σ-algebra $\mathcal{A}$ on X. The elements of $\mathcal{A}$ are called *measurable sets*.

Definition 12.5: A *measure* μ on a set X is a map $\mu : \mathcal{A} \to [0, \infty]$ on a σ-algebra $\mathcal{A}$ of X, such that:

(a) $\mu(\emptyset) = 0$;

(b) If $\{A_i \mid \in \mathbb{N}\}$ is a countable family of mutually disjoint sets in $\mathcal{A}$, meaning that $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

A measure is *finite* if $\mu(X) < \infty$, and *σ-finite* if there is a countable family $\{A_i \in \mathcal{A} \mid i = 1, 2, \ldots\}$ of measurable sets of X such that $\mu(A_1) < \infty$ and

$$X = \bigcup_{i=1}^{\infty} A_i.$$

Example 12.6: Let X be an arbitrary set and $\mathcal{A}$ the σ-algebra consisting of all subsets of X. The *counting measure* ν on X is defined by

$$\nu(A) = \text{ the number of elements of } A,$$

with the convention that if A is an infinite set, then $\nu(A) = \infty$. The counting measure is finite if X is a finite set, and σ-finite if X is countable.

Example 12.7: We define the delta measure δ_{x_0} supported at $x_0 \in \mathbb{R}^n$ on the Borel σ-algebra $\mathcal{R}(\mathbb{R}^n)$ of $\mathbb{R}^n$ by

$$\delta_{x_0}(A) = \begin{cases} 1 & \text{if } x_0 \in A, \\ 0 & \text{if } x_0 \notin A. \end{cases}$$

Theorem 12.10: A subset A of $\mathbb{R}^n$ is Lebesgue measurable if and only if for every $\epsilon > 0$, there is a closed set F and an open set G such that $F \subset A \subset G$ and $\lambda(G \backslash F) < \epsilon$. Moreover,

$$\begin{aligned} \lambda(A) &= \inf\{\lambda(U) \mid U \text{ is open and } U \supset A\} \\ &= \sup\{\lambda(K) \mid K \text{ is compact and } K \subset A\}. \end{aligned}$$

Thus, a Lebesgue measurable set may be approximated from the outside by open sets, and from the inside by compact sets.

Almost Everywhere A property that holds except on a set of measure zero is said to hold *almost everywhere* or *a.e.* for short.

12.2 Measurable Functions

Definition 12.19: Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. A *measurable function* is a mapping $f : X \to Y$ such that

$$f^{-1}(B) \in \mathcal{A} \quad \text{for every } B \in \mathcal{B}.$$

Proposition 12.23: Let $(X, \mathcal{A})$ be a measurable space. A function $f : X \to \overline{\mathbb{R}}$ is measurable if and only if the set $\{x \in X \mid f(x) < c\}$ belongs to $\mathcal{A}$ for every $c \in \mathbb{R}$. In this proposition, the sets $\{f(x) \leq c\}$, $\{f(x) > c\}$, or $\{f(x) \geq c\}$, could be used equally well.

Theorem 12.24: If (f_n) is a sequence of measurable functions that converges pointwise to a function f, then f is measurable. If $(X, \mathcal{A}, \mu)$ is a complete measure space and (f_n) converges pointwise-a.e. to f, then f is measurable.

Definition 12.25: A function $\varphi : X \to \mathbb{R}$ on a measurable space $(X, \mathcal{A})$ is a *simple function* if there are measurable sets $A_1, A_2, \ldots, A_n$ and real numbers $c_1, c_2, \ldots, c_n$ such that

$$\varphi = \sum_{i=1}^{n} c_i \chi_{A_i}.$$

Here, χ_A is the characteristic function of the set A, meaning that

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Theorem 12.26: Let $f : X \to [0, \infty]$ be a nonnegative, measurable function. There is a monotone increasing sequence $\{\varphi_n\}$ of simple function that converges pointwise to f.

12.3 Integration

Definition 12.27: Let $f : X \to [0, \infty]$ be a nonnegative measurable function on a measure space $(X, \mathcal{A}, \mu)$. We define

$$\int f d\mu = \sup \left\{ \int \varphi d\mu \mid \varphi \text{ is simple and } \varphi \leq f \right\}.$$

if $f : X \to \overline{\mathbb{R}}$ and $f = f_+ - f_-$, where f_+ and f_- are the positive and negative parts of f, the we define

$$\int f d\mu = \int f_+ d\mu - \int f_- d\mu,$$

provided that at least one of the integrals on the right had side is finite. If

$$\int |f| d\mu = \int f_+ d\mu + \int f_- d\mu < \infty,$$

then we say that f is *integrable* or *summable.* The Lebesgue integral does not assign a value to the integral of a highly oscillatory function f for which both $\int f_+ d\mu$ and $\int f_- d\mu$ are infinite.

Example 12.29: If δ_{x_0} is the delta-measure, and $f : \mathbb{R}^n \to \mathbb{R}$ is a Borel measurable function, then

$$\int f d\delta_{x_0} = f(x_0).$$

We have $f = g$ a.e. with respect to δ_{x_0} if and only if $f(x_0) = g(x_0)$.

Example 12.30: Let ν be the counting measure on the set $\mathbb{N}$ of natural numbers defined in Example 12.6. If $f : \mathbb{N} \to \mathbb{R}$, then

$$\int f d\nu = \sum_{n=1}^{\infty} f_n,$$

where $f_n = f(n)$

12.4 Convergence Theorems

Theorem 12.33 (Monotone Convergence): Suppose that (f_n) is monotone increasing sequence of non-negative, measurable functions $f_n : X \to [0, \infty]$ on a measurable space $(X, \mathcal{A}, \mu)$. Let $f : X \to [0, \infty]$ be the pointwise limit,

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Then

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

Theorem 12.34 (Fatou): If (f_n) is any sequence of nonnegative measurable functions $f_n : X \to [0, \infty]$ on a measure space $(X, \mathcal{A}, \mu)$, then

$$\int \left(\liminf_{n\to\infty} f_n \right) d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

Theorem 12.35 (Lebesgue Dominated Convergence): Suppose that (f_n) is a sequence of integrable functions, $f_n : X \to \overline{\mathbb{R}}$ on a measure space $(X, \mathcal{A}, \mu)$ that converge to a pointwise limiting function $f : X \to \overline{\mathbb{R}}$. If there is an integrable function $g : X \to [0, \infty]$ such that

$$|f_n(x)| \leq g(x) \quad \text{for all } x \in X \text{ and } n \in \mathbb{N},$$

then f is integrable and

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

12.5 Product Measure and Fubini's Theorem

Definition 12.38: Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. The *product σ-algebra* $\mathcal{A} \otimes \mathcal{B}$ is the σ-algebra on $X \times Y$ that is generated by the collection of sets

$$\{A \times B \mid A \in \mathcal{A},\ B \in \mathcal{B}\}$$

Theorem 12.39: Suppose that $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces. There is a unique product measure $\mu \otimes \nu$, defined on $\mathcal{A} \otimes \mathcal{B}$, with the property that for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$

$$(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B).$$

Theorem 12.41 (Fubini): Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces. Suppose that $f : X \times Y \to \overline{\mathbb{R}}$ is an $(\mathcal{A} \otimes \mathcal{B})$-measurable function.

(a) The function f is integrable, meaning that

$$\int_{X \times Y} |f| d\mu \otimes d\nu < \infty,$$

if and only if either of the following iterated integrals is finite:

$$\int_X \left(\int_Y |f_x(y)| d\nu(y) \right) d\mu(x),$$

$$\int_Y \left(\int_X |f^y(x)| d\mu(x) \right) d\nu(y).$$

(b) If f is integrable, then

$$\int_{X \times Y} f(x,y) d(\mu(x) \otimes \nu(y)) = \int_X \left(\int_Y f_x(y) d\nu(y) \right) d\mu(x),$$

$$\int_{X \times Y} f(x,y) d(\mu(x) \otimes \nu(y)) = \int_Y \left(\int_X f^y(x) d\mu(x) \right) d\nu(y).$$

12.6 The L^p Spaces

Definition 12.45: Let $(X, \mathcal{A}, \mu)$ be a measure space and $1 \leq p < \infty$. The space $L^p(X, \mu)$ is the space of equivalence classes of measurable functions $f : X \to \mathbb{C}$, with respect to the equivalence relation of a.e.-equality, such that

$$\int |f|^p d\mu < \infty.$$

The L^p-norm of f is defined by

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p}.$$

Theorem 12.46: If $(X, \mathcal{A}, \mu)$ is a measure space and $1 \leq p \leq \infty$, then $L^p(X)$ is a Banach space.

Theorem 12.48: Suppose that $(X, \mathcal{A}, \mu)$ is a measure space and $1 \leq p \leq \infty$. If $f \in L^p(X)$, then there is a sequence (φ_n) of simple functions $\varphi_n : X \to \mathbb{C}$ such that

$$\lim_{n \to \infty} \|f - \varphi_n\|_p = 0.$$

Theorem 12.49: If $1 \leq p < \infty$, then $L^p(\mathbb{R}^n)$ is a separable metric space.

Theorem 12.50: The space $C_c(\mathbb{R}^n)$ of continuous functions with compact support is dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

Theorem 12.51: If $1 \leq p < \infty$, then $C_c^\infty(\mathbb{R}^n)$ is a dense subspace of $L^p(\mathbb{R}^n)$.

12.7 The Basic Inequalities

Theorem 12.54 (Hölder): Let $1 \le p, p' \le \infty$ satisfy $1/p + 1/p' = 1$. If $f \in L^p(X, \mu)$ and $g \in L^{p'}(X, \mu)$, then $fg \in L^1(X, \mu)$ and

$$\left| \int fg d\mu \right| \le \|f\|_p \|g\|_{p'}.$$

Proposition 12.55: Suppose that (X, μ) is a finite measure space, meaning that $\mu(X) < \infty$, and $1 \le q \le p \le \infty$. Then

$$L^1(X, \mu) \supset L^q(X, \mu) \supset L^p(X, \mu) \supset L^\infty(X, \mu)$$

Theorem 12.56 (Minkowski): If $1 \le p \le \infty$, and $f, g \in L^p(X, \mu)$, then $f + g \in L^p(X, \mu)$, and

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

Theorem 12.58 (Young): Suppose that $1 \le p, q, r \le \infty$, satisfy

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g \in L^r(\mathbb{R}^n)$, and

$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$

12.8 The Dual Space of L^p

Theorem 12.59: If $1 < p < \infty$, then every $\varphi \in L^p(X)^*$ is of the form

$$\varphi(f) = \int_X fg \, d\mu$$

for some $g \in L^{p'}(X)$, where $1/p + 1/p' = 1$. If μ is σ-finite the same conclusion holds when $p = 1$ and $p' = \infty$. Moreover,

$$\|\varphi\|_{(L^p)} = \|g\|_{L^{p'}}.$$

Definition 12.60: Suppose that $1 \le p < \infty$. A sequence (f_n) *converges weakly* to f in L^p, written $f_n \rightharpoonup f$, if

$$\lim_{n \to \infty} \int f_n g \, d\mu = \int fg \, d\mu \quad \text{for every } g \in L^{p'},$$

where p' is the Hölder conjugate of p. when $p = \infty$ and $p' = 1$, the condition above corresponds to weak-* convergence in L^∞.

Theorem 12.62: Suppose that (f_n) is a bounded sequence in $L^p(X)$, meaning that there is a constant M such that $\|f_n\| \le M$ for every $n \in \mathbb{N}$. if $1 < p < \infty$, then there is a subsequence (f_{n_k}) and a function $f \in L^p(X)$ with $\|f\| \le M$ such that $f_{n_k} \rightharpoonup f$ as $k \to \infty$ weakly in $L^p(X)$.

Other Useful Things to Remember

Proposition 5.30: Let $T : X \to Y$ be a bounded linear map between Banach spaces X, Y. The following statements are equivalent:

(a) There is a constant $c > 0$ such that

$$c\|x\| \leq \|Tx\| \quad \text{for all } x \in X;$$

(b) T has closed range, and the only solution of the equation $Tx = 0$ is $x = 0$

Proposition 5.43: Let X, Y, Z be Banach spaces.

(a) If $S, T \in \mathcal{B}(X, Y)$ are compact, then any linear combination of S and T is compact.

(b) If (T_n) is a sequence of compact operator in $\mathcal{B}(X, Y)$ converging uniformly to T, then T is compact.

(c) If $T \in \mathcal{B}(X, Y)$ has finite-dimensional range, Then T is compact.

(d) Let $S \in \mathcal{B}(X, Y)$, $T \in \mathcal{B}(Y, Z)$. If S is bounded and T is compact, or S is compact and T is bounded, then $TS \in \mathcal{B}(X, Z)$ is compact.