Chapter 7: Fourier Series

7.1 The Fourier Basis

The Torus (T): A 2π-periodic function on \( \mathbb{R} \) may be identified with a function on a circle, or one-dimensional torus, \( T = \mathbb{R}/(2\pi \mathbb{Z}) \). which we define by identifying points in \( T \) that differ by \( 2\pi n \) for some \( n \in \mathbb{Z} \).

Inner Product (\( L^2(T) \)):

\[
\langle f, g \rangle = \int_T f(x)g(x)dx
\]

Fourier Basis: The Fourier basis elements are the functions

\[
e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}.
\]

Fourier Series: Any function \( f \in L^2(T) \) may be expanded in a Fourier series as

\[
f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx},
\]

where the equality means convergence of the partial sums of \( f \) in the \( L^2 \)-norm, and

\[
\hat{f}_n = \langle e_n, f \rangle = \frac{1}{\sqrt{2\pi}} \int_T f(x)e^{-inx}dx.
\]

Definition 7.1: A family of functions \( \{ \varphi_n \in C(T) | n \in \mathbb{N} \} \) is an approximate identity if:

(a) \( \varphi_n(x) \geq 0 \);

(b) \( \int_T \varphi_n(x)dx = 1 \) for every \( n \in \mathbb{N} \);

(c) \( \lim_{n \to \infty} \int_{|x| \leq \delta} \varphi_n(x)dx = 0 \) for every \( \delta > 0 \).
Theorem 7.2: If \( \{ \varphi_n \in C(T) \mid n \in \mathbb{N} \} \) is an approximate identity and \( f \in C(T) \), then \( \varphi_n \ast f \) converges uniformly to \( f \) as \( n \to \infty \).

Theorem 7.3: The trigonometric polynomials are dense in \( C(T) \) with respect to the uniform norm.

Proposition 7.4: If \( f, g \in L^2(T) \), then \( f \ast g \in C(T) \) and
\[
\|f \ast g\|_\infty \leq \|f\|_2 \|g\|_2.
\]

Theorem 7.5 (Convolution): If \( f, g \in L^2(T) \), then
\[
(\hat{f} \ast \hat{g})_n = \sqrt{2\pi} \hat{f}_n \hat{g}_n.
\]

### 7.2 Fourier Series of Differentiable Functions

**Definition 7.6:** The Sobolev space \( H^1(T) \) consists of all functions
\[
f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx} \in L^2(T)
\]
such that
\[
\sum_{n=-\infty}^{\infty} n^2 |\hat{f}_n|^2 < \infty.
\]
The weak \( L^2 \)-derivative \( f' \in L^2(T) \) of \( f \in H^1(T) \) is defined by the \( L^2 \)-convergent Fourier series
\[
f'(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} in\hat{f}_n e^{inx} \in L^2(T).
\]
More generally
\[
H^k(T) = \left\{ f \in L^2(T) \mid f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \sum_{n=-\infty}^{\infty} |n|^{2k} |c_n|^2 < \infty \right\},
\]
for \( k \geq 0 \).

**Definition 7.7:** A function \( f \in L^2(T) \) belongs to \( H^1(T) \) if there is a constant \( M \) such that
\[
\left| \int_T f' \varphi dx \right| \leq M \| \varphi \|_{L^2} \quad \text{for all } \varphi \in C^1(T)
\]
for \( f \in H^1(T) \), then the weak derivative \( f' \) of \( f \) is the unique element of \( L^2(T) \) such that
\[
\int_T f' \varphi dx = -\int_T f \varphi' dx \quad \text{for all } \varphi \in C^1(T).
\]

**Lemma 7.8:** Suppose that \( f \in H^k(T) \) for \( k > 1/2 \). Let
\[
S_N(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} \hat{f}_n e^{inx}
\]
be the $N$th partial sum of the Fourier series of $f$, and define

$$
\| f^{(k)} \| = \left( \sum_{n=-\infty}^{\infty} |n|^{2k} |\hat{f}_n|^2 \right)^{\frac{1}{2}}.
$$

Then there is a constant $C_k$, independent of $f$, such that

$$
\| S_N - f \|_\infty \leq \frac{C_k}{N^{k-\frac{1}{2}}} \| f^{(k)} \|,
$$

and $(S_N)$ converges uniformly to $f$ as $N \to \infty$.

**Theorem 7.9:** If $f \in H^k(\mathbb{T})$ for $k > 1/2$, then $f \in C(\mathbb{T})$. More generally: If $f \in H^k(\mathbb{T}^d)$ and $k > j + d/2$ then $f \in C^j(\mathbb{T}^d)$. 
Chapter 8: Bounded Linear Operators on a Hilbert Space

8.1 Orthogonal Projections

Definition 8.1: A projection on a linear space \( X \) is a linear map \( P : X \to X \) such that \( P^2 = P \).

Theorem 8.2: Let \( X \) be a linear space.
(a) If \( P : X \to X \) is a projection, then \( X = \text{ran} \, P \oplus \ker P \).
(b) If \( X = M \oplus N \), where \( M \) and \( N \) are linear subspaces of \( X \), then there is a projection \( P : X \to X \) with \( \text{ran} \, P = M \) and \( \ker P = N \).

Definition 8.3: An orthogonal projection on a Hilbert space \( \mathcal{H} \) is a linear map \( P : \mathcal{H} \to \mathcal{H} \) that satisfies
\[
P^2 = P, \quad \langle Px, y \rangle = \langle x, Py \rangle \quad \text{for all } x, y \in \mathcal{H}.
\]
An orthogonal projection is necessarily bounded.

Proposition 8.4: If \( P \) is a nonzero orthogonal projection, then \( \| P \| = 1 \).

Theorem 8.5: Let \( \mathcal{H} \) be a Hilbert space.
(a) If \( P \) is an orthogonal projection on \( \mathcal{H} \), then \( \text{ran} \, P \) is closed, and
\[
\mathcal{H} = \text{ran} \, P \oplus \ker P
\]
is the orthogonal direct sum of \( \text{ran} \, P \) and \( \ker P \).
(b) If \( M \) is a closed subspace of \( \mathcal{H} \), then there is an orthogonal projection \( P \) on \( \mathcal{H} \) with \( \text{ran} \, P = M \) and \( \ker P = M^\perp \).

Even and Odd Projections: The space \( L^2(\mathbb{R}) \) is the orthogonal direct sum of the space \( M \) of even functions and the space \( N \) of the odd functions. The orthogonal projections \( P \) and \( Q \) of \( \mathcal{H} \) onto \( M \) and \( N \), respectively, are given by
\[
Pf(x) = \frac{f(x) + f(-x)}{2}, \quad Qf(x) = \frac{f(x) - f(-x)}{2}.
\]

note that \( I - P = Q \).

8.2 The Dual of a Hilbert Space

Linear Functional: A linear functional on a complex Hilbert space \( \mathcal{H} \) is a linear map from \( \mathcal{H} \) to \( \mathbb{C} \). A linear function \( \varphi \) is bounded, or continuous, if there exist a constant \( M \) such that
\[
|\varphi(x)| \leq M \| x \| \quad \text{for all } x \in \mathcal{H}.
\]
The norm of a bounded linear function \( \varphi \) is
\[
\| \varphi \| = \sup_{\| x \| = 1} |\varphi(x)|.
\]
if $y \in \mathcal{H}$, then
\[ \varphi_y(x) = \langle y, x \rangle \]
is a bounded linear functional on $\mathcal{H}$, with $\|\varphi_y\| = \|y\|$. 

**Theorem 8.12 (Riesz representation):** If $\varphi$ is a bounded linear functional on a Hilbert space $\mathcal{H}$, then there is a unique vector $y \in \mathcal{H}$ such that
\[ \varphi(x) = \langle y, x \rangle \quad \text{for all } x, y \in \mathcal{H}. \]

### 8.3 The Adjoint of an Operator

**Adjoint Operator:** Let $A \in \mathcal{B}(\mathcal{H})$ then there exist an unique $A^* \in \mathcal{B}(\mathcal{H})$ (known as the adjoint) such that
\[ \langle x, Ay \rangle = \langle A^*x, y \rangle \quad \text{for all } x, y \in \mathcal{H}. \]

**Left and Right Shift Operators:** Suppose that $S$ and $T$ are the right and left shift operators on the sequence space $\ell^2(\mathbb{N})$, defined by
\[ S(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots), \quad T(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots). \]
Then $T = S^*$.

**Theorem 8.17:** If $A : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator, then
\[ \overline{\text{ran} A} = (\text{ker} A^*)^\perp, \quad \text{ker} A = (\text{ran} A^*)^\perp. \]

**Theorem 8.18:** Suppose that $A : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on a Hilbert space $\mathcal{H}$ with closed range. Then the equation $Ax = y$ has a solution for $x$ if and only if $y$ is orthogonal to $\text{ker} A^*$.

### 8.4 Self-Adjoint and Unitary Operators

**Definition 8.23:** A bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is **self-adjoint** if $A^* = A$. Equivalently, a bounded linear operator $A$ on $\mathcal{H}$ is self-adjoint if and only if
\[ \langle x, Ay \rangle = \langle Ax, y \rangle \quad \text{for all } x, y \in \mathcal{H}. \]

**Nonnegative, Positive/Positive Definite, Coercive:** Let $A$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then $A$ is
- **nonnegative** if $\langle x, Ax \rangle \geq 0$ for all $x \in \mathcal{H}$,
- **positive** or **positive definite** if $\langle x, Ax \rangle > 0$ for all nonzero $x \in \mathcal{H}$,
- **coercive** if there exists a $c > 0$ such that $\langle x, Ax \rangle \geq c\|x\|^2$ for all $x \in \mathcal{H}$. 

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Lemma 8.26: If $A$ is a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$, then
\[ \|A\| = \sup_{\|x\|=1} |\langle x, Ax \rangle|. \]

Corollary 8.27: If $A$ is a bounded self-adjoint operator on a Hilbert space then $\|A^*A\| = \|A\|^2$. If $A$ is self-adjoint, then $\|A^2\| = \|A\|^2$.

Definition 8.28: A linear map $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ between real or complex Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ it is said to be orthogonal or unitary, respectively, if it is invertible and if
\[ \langle Ux, Uy \rangle_{\mathcal{H}_2} = \langle x, y \rangle_{\mathcal{H}_1} \text{ for all } x, y \in \mathcal{H}_1. \]
Two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic as Hilbert spaces if there is a unitary linear map between them. If $U : \mathcal{H} \rightarrow \mathcal{H}$ (i.e $\mathcal{H}_1 = \mathcal{H}_2$), then $U$ is unitary if and only if $U^*U = UU^* = I$.

Example 8.30: If $A$ is a bounded self-adjoint operator, then
\[ e^{iA} = \sum_{n=0}^{\infty} \frac{1}{n!}(iA)^n \]
is unitary.

8.6 Weak Convergence in a Hilbert Space

Weak Convergence: A sequence $(x_n)$ in a Hilbert space $\mathcal{H}$ converges weakly to $x \in \mathcal{H}$ if
\[ \lim_{n \to \infty} \langle x_n, y \rangle = \langle x, y \rangle \text{ for all } y \in \mathcal{H}. \]

Theorem 8.39 (Uniform Boundedness): Suppose that
\[ \{\varphi_n : X \rightarrow \mathbb{C}|n \in \mathbb{N}\} \]
is a set of linear functionals on a Banach Space $X$ such that the set of complex numbers $\{\varphi_n(x)|n \in \mathbb{N}\}$ is bounded for each $x \in X$. Then $\{\|\varphi_n\| |n \in \mathbb{N}\}$ is bounded.

Theorem 8.40: Suppose that $(x_n)$ is a sequence in a Hilbert space $\mathcal{H}$ and $D$ is a dense subset of $\mathcal{H}$. Then $(x_n)$ converges weakly to $x$ if and only if:

(a) $\|x_n\| \leq M$ for some constant $M$;
(b) $\langle x_n, y \rangle \rightarrow \langle x, y \rangle$ as $n \to \infty$ for all $y \in D$.

Note: this is extremely useful for showing weak convergence by letting $D$ be a basis for $\mathcal{H}$ as in Example 8.41.

Proposition 8.44: If $(x_n)$ converges weakly to $x$, then
\[ \|x\| \leq \liminf_{n \to \infty} \|x_n\|. \]
If, in addition,
\[ \lim_{n \to \infty} \|x_n\| = \|x\|, \]
then $(x_n)$ converges strongly to $x$. 
**Theorem 8.45 (Banach-Alaoglu):** The closed unit ball of a Hilbert space is weakly compact.

**Definition 8.47:** Let $f : C \to \mathbb{R}$ be a real-valued function on a convex subset $C$ of a real or complex linear space. Then $f$ is *convex* if

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

for all $x, y \in C$ and $0 \leq t \leq 1$. If we have a strict inequality in this equation whenever $x \neq y$ and $0 < t < 1$, then $f$ is *strictly convex*.

**Theorem 8.48 (Mazur):** If $(x_n)$ converges weakly to $x$ in a Hilbert space, then there is a sequence $(y_n)$ of finite convex combinations of $\{x_n\}$ such that $(y_n)$ converges strongly to $x$. 


Chapter 9: Spectrum of Bounded Linear Operators

9.2 The Spectrum

A Quick Note: A great reference for the spectrum of common operators (such as general multiplication operators and the shift operators) check out this wonderful pdf. http://www.math.ubc.ca/~feldman/m511/spectralExamples.pdf

Definition 9.3: The resolvent set of an operator \( A \in \mathcal{B}(\mathcal{H}) \), denoted by \( \rho(A) \), is the set of complex numbers \( \lambda \) such that \( (A - \lambda I) : \mathcal{H} \to \mathcal{H} \) is one-to-one and onto. The spectrum of \( A \), denoted by \( \sigma(A) \), is the complement of the resolvent set in \( \mathbb{C} \), meaning that \( \sigma(A) = \mathbb{C} \setminus \rho(A) \).

Definition 9.4: Suppose that \( A \) is a bounded linear operator on a Hilbert space \( \mathcal{H} \).

(a) The point spectrum of \( A \) consists of all \( \lambda \in \sigma(A) \) such that \( A - \lambda I \) is not one-to-one. In this case \( \lambda \) is called an eigenvalue of \( A \).

(b) The continuous spectrum of \( A \) consists of all \( \lambda \in \sigma(A) \) such that \( A - \lambda I \) is one-to-one but not onto, and \( \text{ran}(A - \lambda I) \) is dense in \( \mathcal{H} \).

(c) The residual spectrum of \( A \) consists of all \( \lambda \in \sigma(A) \) such that \( A - \lambda I \) is one-to-one but not onto, and \( \text{ran}(A - \lambda I) \) is not dense in \( \mathcal{H} \).

Proposition 9.6: If \( A \) is a bounded linear operator on a Hilbert space, then the resolvent set \( \rho(A) \) is an open subset of \( \mathbb{C} \) that contains the exterior disc \( \{ \lambda \in \mathbb{C} \mid |\lambda| > \|A\| \} \). The resolvent \( R_\lambda \) is an operator-valued analytic function of \( \lambda \) defined on \( \rho(A) \).

Proposition 9.7: Let the spectral radius be defined as

\[
r(A) = \sup \{|\lambda| \mid \lambda \in \sigma(A)\}.
\]

If \( A \) is a bounded linear operator, then

\[
r(A) = \lim_{n \to \infty} \|A^n\|^{1/n}.
\]

If \( A \) is self-adjoint, then \( r(A) = \|A\| \).

Proposition 9.8: The spectrum of a bounded operator on a Hilbert space is nonempty.

The Spectral Theorem for Compact, Self-Adjoint Operators

Lemma 9.9: The eigenvalues of a bounded, self-adjoint operator are real, and the eigenvectors associated with different eigenvalues are orthogonal.

Invariant Subspace: A linear subspace \( \mathcal{M} \) of \( \mathcal{H} \) is called an invariant subspace of a linear operator \( A \) on \( \mathcal{H} \) if \( Ax \in \mathcal{M} \) for all \( x \in \mathcal{M} \).

Lemma 9.11: If \( A \) is a bounded, self-adjoint operator on a Hilbert space \( \mathcal{H} \) and \( \mathcal{M} \) is an invariant subspace of \( A \), then \( \mathcal{M}^\perp \) is an invariant subspace of \( A \).

Proposition 9.12: If \( \lambda \) belongs to the residual spectrum of a bounded operator \( A \) on a Hilbert space, then \( \lambda \) is an eigenvalue of \( A^* \).

Lemma 9.13: If \( A \) is a bounded, self-adjoint operator on a Hilbert space, then the spectrum of \( A \) is real and is contained in the interval \([-\|A\|, \|A\|]\).
Corollary 9.14: The residual spectrum of a bounded, self-adjoint operator is empty.

Proposition 9.15: A nonzero eigenvalue of a compact, self-adjoint operator has finite multiplicity. A countably infinite set of nonzero eigenvalues has zero as an accumulation point, and no other accumulation points.

Theorem 9.16 (Spectral Theorem for Compact, Self-Adjoint Operators): Let $A : \mathcal{H} \to \mathcal{H}$ be a compact, self-adjoint operator on a Hilbert space $\mathcal{H}$. There is an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of $A$. The nonzero eigenvalue of $A$ form a finite or countably infinite set $\{\lambda_k\}$ of real numbers, and

$$A = \sum_k \lambda_k P_k,$$

where $P_k$ is the orthogonal projection onto the finite-dimensional eigenspace of eigenvectors with eigenvalues $\lambda_k$. If the number of nonzero eigenvalues is countably infinite then the series above converges to $A$ in operator norm.

9.4 Compact Operators

Theorem 9.17: Let $E$ be a subset of an infinite-dimensional, separable Hilbert space $\mathcal{H}$.

(a) If $E$ is precompact, then for every orthonormal set $\{e_n \mid n \in \mathbb{N}\}$ and every $\epsilon > 0$, there is an $N$ such that

$$\sum_{n=N+1}^{\infty} |\langle e_n, x \rangle|^2 < \epsilon$$

for all $x \in E$.

(b) if $E$ is bounded and there is an orthonormal basis $\{e_n\}$ of $\mathcal{H}$ with the property that for every $\epsilon > 0$ there is an $N$ such that the sum in (a) holds, then $E$ is precompact.

Example 9.19: The diagonal operator $A : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by

$$A(x_1, x_2, x_3, \ldots, x_n, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n, \ldots),$$

where $\lambda_n \in \mathbb{C}$ is compact if and only if $\lambda_n \to 0$ as $n \to \infty$. Any compact, normal operator on a separable Hilbert space is unitarily equivalent to such a diagonal operator. Note: this implies that the identity operator on a Hilbert space $\mathcal{H}$ is compact if and only if $\mathcal{H}$ is finite-dimensional.

Definition 9.20: A bounded linear operator $A$ on a separable Hilbert space $\mathcal{H}$ is Hilbert-Schmidt if there is an orthonormal basis $\{e_n \mid n \in \mathbb{N}\}$ such that

$$\sum_{n=1}^{\infty} \|A e_n\|^2 < \infty.$$

If $A$ is a Hilbert-Schmidt operator, then

$$\|A\|_{HS} = \sqrt{\sum_{n=1}^{\infty} \|A e_n\|^2}$$

is called the Hilbert-Schmidt norm of $A$.

Theorem 9.21: A Hilbert-Schmidt operator is compact.

Theorem 9.24: A bounded linear operator on a Hilbert space is compact if and only if it maps weakly convergent sequences into strongly convergent sequences.
9.5 Functions of Operators

Theorem 9.25 (Spectral Mapping): If $A$ is a compact, self-adjoint operator on a Hilbert space and $f : \sigma(A) \to \mathbb{C}$ is continuous, then

$$\sigma(f(A)) = f(\sigma(A)).$$

Here $\sigma(f(A))$ is the spectrum of $f(A)$, and $f(\sigma(A))$ is the image of the spectrum of $A$ under $f$,

$$f(\sigma(A)) = \{ \mu \in \mathbb{C} \mid \mu = f(\mu) \text{ for some } \lambda \in \sigma(A) \}.$$
Chapter 11: Distributions and the Fourier Transform

11.1 The Schwartz Space

Multi-Index: A multi-index

\[ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n \quad \text{with} \quad \mathbb{Z}_+ = \{ n \in \mathbb{Z} \mid n \geq 0 \} \]

is an \( n \)-tuple of nonnegative integers \( a_i \geq 0 \). For multi-indices \( \alpha, \beta \in \mathbb{Z}_+^n \) and \( x \in \mathbb{R}^n \), we define

(a) \( |\alpha| = \sum_{i=1}^{n} \alpha_i \),
(b) \( \alpha! = \prod_{i=1}^{n} \alpha_i! \),
(c) \( \alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n) \),
(d) \( \alpha \geq \beta \) if and only if \( \alpha_i \geq \beta_i \) for \( i = 1, \ldots, n \),
(e) \( \partial^\alpha = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n} \),
(f) \( x^\alpha = \prod_{i=1}^{n} x_i^{\alpha_i} \),
(g) \( |x| = \sqrt{x_1^2 + \cdots + x_n^2} \).

Leibnitz Rule: The Leibnitz rule for the derivative of the product of \( f, g \in C^k(\mathbb{R}^n) \) may be written as

\[ \partial^\alpha (fg) = \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta!\gamma!} (\partial^\beta f)(\partial^\gamma g). \]

Definition 11.1: The Schwartz space \( \mathcal{S}(\mathbb{R}^n) \), or \( \mathcal{S} \) for short, consists of all functions \( \varphi \in C^\infty(\mathbb{R}^n) \) such that

\[ p_{\alpha,\beta}(\varphi) = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)| \]

is finite for every pair of multi-indices \( \alpha, \beta \in \mathbb{Z}_+^n \). If \( \varphi \in \mathcal{S} \), then for every \( d \in \mathbb{N} \) and \( \alpha \in \mathbb{Z}_+^n \) there is a constant \( C_{d,\alpha} \) such that

\[ |\partial^\alpha \varphi(x)| \leq \frac{C_{d,\alpha}}{(1 + |x|^2)^{d/2}} \quad \text{for all} \quad x \in \mathbb{R}^n. \]

Thus an element of \( \mathcal{S} \) is a smooth function such that the function and all of its derivatives decay faster than any polynomial as \( |x| \to \infty \). Elements of \( \mathcal{S} \) are called Schwartz functions, or test functions.

Definition 11.2: Suppose that \( X \) is a real or complex linear space. A function \( p : X \to \mathbb{R} \) is a seminorm on \( X \) if it has the following properties:

(a) \( p(x) \geq 0 \) for all \( x \in X \);
(b) \( p(x + y) \leq p(x) + p(y) \) for all \( x, y \in X \);
(c) \( p(\lambda x) = |\lambda| p(x) \) for every \( x \in X \) and \( \lambda \in \mathbb{C} \).

A seminorm has the same properties as a norm, except that \( p(x) = 0 \) does not need to imply \( x = 0 \). If a family of seminorms \( \{p_1, \ldots, p_n\} \) is finite and separates points, then

\[ \|x\| = p_1(x) + \cdots + p_n(x) \]

defines a norm on \( X \).

Proposition 11.3: The Schwartz space \( \mathcal{S} \) with the metrizable topology generated by the countable family of seminorms

\[ \{p_{\alpha,\beta} \mid \alpha, \beta \in \mathbb{Z}_+^n \}, \]
where \( p_{\alpha,\beta} \) is given by
\[
\| \varphi \|_{\alpha,\beta} = p_{\alpha,\beta}(\varphi) = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)|,
\]
is complete.

**Proposition 11.4:** For each \( \alpha \in \mathbb{Z}_n^+ \), the partial differentiation operator \( \partial^\alpha : S \to S \) is continuous linear operator on \( S \).

### 11.2 Tempered Distributions

**Tempered Distributions:** The topological dual space of \( S \), denoted by \( S^* \) or \( S' \), is the space of continuous linear functionals \( T : S \to \mathbb{C} \). Elements of \( S^* \) are called *tempered distributions*. Since \( S \) is a metric space, a functional \( T : S \to \mathbb{C} \) is continuous if and only if for every convergent sequence \( \varphi_n \to \varphi \) in \( S \), we have
\[
\lim_{n \to \infty} T(\varphi_n) = T(\varphi).
\]
The continuity of a linear functional \( T \) is implied by an estimate of the form
\[
|T(\varphi)| \leq \sum_{|\alpha|,|\beta| \leq d} c_{\alpha,\beta} \| \varphi \|_{\alpha,\beta}
\]
for some \( d \in \mathbb{Z}_+ \) and constants \( c_{\alpha,\beta} \geq 0 \).

**Example 11.5:** The fundamental example of a distribution is the *delta function*. We define \( \delta : S \to \mathbb{C} \) by evaluation at 0:
\[
\delta(\varphi) = \varphi(0).
\]

### 11.3 Operations on Distributions

**Example 11.9:** If \( T = \delta \) is the delta function, then
\[
\langle f\delta, \varphi \rangle = \langle \delta, f\varphi \rangle = f(0)\varphi(0) = \langle f(0)\delta, \varphi \rangle.
\]
Hence, \( f\delta = f(0)\delta \).

**Definition 11.10:** Suppose that \( T \) is a tempered distribution and \( \alpha \) is a multi-index. the \( \alpha \)th *distributional derivative* of \( T \) is the tempered distribution \( \partial^\alpha T \) defined by
\[
\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle \quad \text{for all } \varphi \in S.
\]

**Theorem 11.11:** For every \( T \in S^* \) there is a continuous function \( f : \mathbb{R}^n \to \mathbb{C} \) of polynomial growth and a multi-index \( \alpha \in \mathbb{Z}_n^+ \) such that \( T = \partial^\alpha f \).

**Example 11.14:** The derivative of the one-dimensional delta function \( \delta \) is given by
\[
\langle \delta', \varphi \rangle = -\langle \delta, \varphi' \rangle = -\varphi'(0).
\]
More generally, the \( k \)th distributional derivative of \( \delta \) is given by
\[
\langle \delta^{(k)}, \varphi \rangle = (-1)^k \varphi^{(k)}(0).
\]
Example 11.16: For each $h \in \mathbb{R}^n$, we define the translation operator $\tau_h : \mathcal{S} \to \mathcal{S}$ by

$$\tau_h f(x) = f(x - h).$$

We therefore define the translation $\tau_h T$ of a distribution $T$ by

$$\langle \tau_h T, \varphi \rangle = \langle T, \tau_{-h} \varphi \rangle \quad \text{for all } \varphi \in \mathcal{S}.$$ 

For instance, we have $\delta_{x_0} = \tau_{x_0} \delta$.

Example 11.17: Let $R : \mathcal{S} \to \mathcal{S}$ be the reflection operator,

$$Rf(x) = f(-x).$$

Thus, for $T \in \mathcal{S}^*$, we defined the reflection $RT \in \mathcal{S}^*$ by

$$\langle RT, \varphi \rangle = \langle T, R\varphi \rangle \quad \text{for all } \varphi \in \mathcal{S}.$$ 

Proposition 11.18: For any $\varphi, \psi, \omega \in \mathcal{S}$, we have:

(a) $\varphi * \psi = \psi * \varphi$,
(b) $(\varphi * \psi) * \omega = \varphi * (\psi * \omega)$,
(c) $\tau_h (\varphi * \psi) = (\tau_h \varphi) * \psi = \varphi * (\tau_h \psi)$ for every $h \in \mathbb{R}^n$.

11.4 The Convergence of Distributions

Convergence in Distribution Space: Let $(T_n)$ be a sequence in $\mathcal{S}^*$. We say that $(T_n)$ converges to $T \in \mathcal{S}^*$ if and only if

$$\lim_{n \to \infty} \langle T_n, \varphi \rangle = \langle T, \varphi \rangle \quad \text{for every } \varphi \in \mathcal{S}.$$ 

We denote convergence is the space of distributions by $T_n \rightharpoonup T$ as $n \to \infty$.

Proposition 11.22: For $n \in \mathbb{N}$, let

$$\sigma_n(x) = \frac{\sin(nx)}{\pi x}.$$ 

Then $\sigma_n \rightharpoonup \delta$ in $\mathcal{S}^*$ as $n \to \infty$.

Theorem 11.23: The Schwartz space is dense in the space of tempered distributions.

11.5 The Fourier Transform of Test Functions

Definition 11.24: If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then the Fourier transform $\hat{\varphi} : \mathbb{R}^n \to \mathbb{C}$ is the function defined by

$$\hat{\varphi}(k) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(x) e^{-ik \cdot x} \, dx \quad \text{for } k \in \mathbb{R}^n.$$ 

We define the Fourier transform operator $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ by $\mathcal{F}\varphi = \hat{\varphi}$.

Eigenvalues of The Fourier Transform: Knowing that $\mathcal{F}^2 \varphi = R \varphi$ where $R$ is a reflection. It is easy to see that $\mathcal{F}^4 \varphi = \varphi$ leading to the characteristic equation $\lambda^4 = 1$. It turns out that the eigenvalues of $\mathcal{F}$ are

$$\sigma_p(\mathcal{F}) = \{ \pm 1, \pm i \}.$$ 

Proposition 11.25: If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then:
(a) \( \hat{\varphi} \in C^\infty(\mathbb{R}^n) \), and 
\[ \partial^\alpha \hat{\varphi} = \mathcal{F}[(-ix)^\alpha \varphi]; \]
(b) \( k^\alpha \hat{\varphi} \) is bounded for every multi-index \( \alpha \in \mathbb{Z}^n_+ \), and 
\[ (ik)^\alpha \hat{\varphi} = \mathcal{F}[\partial^\alpha \varphi]. \]

The Fourier transform \( \mathcal{F}: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \) is a continuous linear map on \( \mathcal{S}(\mathbb{R}^n) \).

**Proposition 11.27:** If \( \varphi, \psi \in \mathcal{S} \) and \( h \in \mathbb{R}^n \), then:
\[
\hat{\tau}_h \hat{\varphi} = e^{-ik \cdot h} \hat{\varphi}, \\
e^{ix \cdot h} \varphi = \tau_h \hat{\varphi}, \\
\hat{\varphi} \ast \psi = (2\pi)^{n/2} \hat{\varphi} \hat{\psi}.
\]

**Definition 11.28:** If \( \varphi \in \mathcal{S} \), then the inverse Fourier transform \( \hat{\varphi} \) is given by
\[
\hat{\varphi}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ik \cdot x} \varphi(k) \, dk.
\]
We define \( \mathcal{F}^*: \mathcal{S} \to \mathcal{S}^* \) by \( \mathcal{F}^* \varphi = \hat{\varphi} \).

**Proposition 11.29:** The map \( \mathcal{F}^* \) is a continuous linear transformation on \( \mathcal{S} \), and \( \mathcal{F}^* = \mathcal{F}^{-1} \).

### 11.6 The Fourier Transform of Tempered Distributions

**Definition 11.30:** The Fourier transform of a tempered distribution \( T \) is the tempered distribution \( \hat{T} = \mathcal{F}T \) defined by
\[
\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle \quad \text{for all} \quad \varphi \in \mathcal{S}.
\]
The inverse Fourier transform \( \check{T} = \mathcal{F}^{-1}T \) on \( \mathcal{S}^* \) is defined by
\[
\langle \check{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle \quad \text{for all} \quad \varphi \in \mathcal{S}.
\]
The map \( \mathcal{F}: \mathcal{S}^* \to \mathcal{S}^* \) is a continuous, one-to-one transformation of \( \mathcal{S}^* \) onto itself.

**Example 11.31:** The Fourier transform of the delta function is
\[
\langle \delta, \varphi \rangle = \langle \delta, \hat{\varphi} \rangle = \hat{\varphi}(0) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(x) \, dx = \frac{1}{(2\pi)^{n/2}} \langle 1, \varphi \rangle.
\]
Hence, the Fourier transform of the delta function is
\[
\hat{\delta} = \frac{1}{(2\pi)^{n/2}}.
\]
11.7 The Fourier Transform of $L^1$

Convergence of the Fourier Integral: The Fourier integral (Fourier transform on $L^1$-functions)

$$\hat{f}(k) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)e^{-ik \cdot x} dx$$

converges if and only if $f \in L^1(\mathbb{R}^n)$, meaning that

$$\int_{\mathbb{R}^n} |f(x)| dx < \infty.$$

Theorem 11.34 (Riemann-Lebesgue): If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$, and

$$(2\pi)^{n/2} \|\hat{f}\|_\infty \leq \|f\|_1.$$ 

Theorem 11.35 (Convolution): If $f, g \in L^1(\mathbb{R}^n)$, then $f * g \in L^1(\mathbb{R}^n)$ and

$$\hat{f * g} = (2\pi)^{n/2} \hat{f} \hat{g}.$$ 

11.8 The Fourier Transform of $L^2$

Theorem 11.37 (Plancherel): The Fourier Transform $\mathcal{F} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is a unitary map. For every $f, g \in L^2(\mathbb{R}^n)$, we have

$$(\hat{f}, \hat{g}) = (f, g),$$

where

$$(f, g) = \int_{\mathbb{R}^n} \overline{f(x)} g(x) dx$$

and $\hat{f} = \mathcal{F} f$. In particular,

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \int_{\mathbb{R}^n} |\hat{f}(k)|^2 dk.$$

Definition 11.38: Let $s \in \mathbb{R}$. The Sobolev space $H^s(\mathbb{R}^n)$ consists of all distributions $f \in \mathcal{S}^*$ whose Fourier transform $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{C}$ is a regular distribution and

$$\int_{\mathbb{R}^n} (1 + |k|^2)^s |\hat{f}(k)|^2 dk < \infty.$$
Chapter 12: Measure Theory and Function Spaces

12.1 Measures

Definition 12.1: A $\sigma$-algebra on a set $X$ is a collection of $\mathcal{A}$ of subsets of $X$ such that:

(a) $\emptyset \in \mathcal{A}$;

(b) If $A \in \mathcal{A}$, then $A^c = X \setminus A \in \mathcal{A}$;

(c) If $\{A_i \mid i \in \mathbb{N}\}$ is a countable family of sets in $\mathcal{A}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

A measurable space $(X, \mathcal{A})$ is a set $X$ and a $\sigma$-algebra $\mathcal{A}$ on $X$. The elements of $\mathcal{A}$ are called measurable sets.

Definition 12.5: A measure $\mu$ on a set $X$ is a map $\mu : \mathcal{A} \to [0, \infty]$ on a $\sigma$-algebra $\mathcal{A}$ of $X$, such that:

(a) $\mu(\emptyset) = 0$;

(b) If $\{A_i \mid i \in \mathbb{N}\}$ is a countable family of mutually disjoint sets in $\mathcal{A}$, meaning that $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

A measure is finite if $\mu(X) < \infty$, and $\sigma$-finite if there is a countable family $\{A_i \in \mathcal{A} \mid i = 1, 2, \ldots\}$ of measurable sets of $X$ such that $\mu(A_1) < \infty$ and

$$X = \bigcup_{i=1}^{\infty} A_i.$$

Example 12.6: Let $X$ be an arbitrary set and $\mathcal{A}$ the $\sigma$-algebra consisting of all subsets of $X$. The counting measure $\nu$ on $X$ is defined by

$$\nu(A) = \text{the number of elements of } A,$$

with the convention that if $A$ is an infinite set, then $\nu(A) = \infty$. The counting measure is finite if $X$ is a finite set, and $\sigma$-finite if $X$ is countable.

Example 12.7: We define the delta measure $\delta_{x_0}$ supported at $x_0 \in \mathbb{R}^n$ on the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R}^n)$ of $\mathbb{R}^n$ by

$$\delta_{x_0}(A) = \begin{cases} 1 & \text{if } x_0 \in A, \\ 0 & \text{if } x_0 \notin A. \end{cases}$$

Theorem 12.10: A subset $A$ of $\mathbb{R}^n$ is Lebesgue measurable if and only if for every $\epsilon > 0$, there is a closed set $F$ and an open set $G$ such that $F \subset A \subset G$ and $\lambda(G \setminus F) < \epsilon$. Moreover,

$$\lambda(A) = \inf\{\lambda(U) \mid U \text{ is open and } U \supset A\} = \sup\{\lambda(K) \mid K \text{ is compact and } K \subset A\}.$$

Thus, a Lebesgue measurable set may be approximated from the outside by open sets, and from the inside by compact sets.

Almost Everywhere A property that holds except on a set of measure zero is said to hold almost everywhere or a.e. for short.
12.2 Measurable Functions

**Definition 12.19:** Let \((X, \mathcal{A})\) and \((Y, \mathcal{B})\) be measurable spaces. A **measurable function** is a mapping \(f : X \to Y\) such that

\[ f^{-1}(B) \in \mathcal{A} \quad \text{for every } B \in \mathcal{B}. \]

**Proposition 12.23:** Let \((X, \mathcal{A})\) be a measurable space. A function \(f : X \to \mathbb{R}\) is measurable if and only if the set \(\{x \in X \mid f(x) < c\}\) belongs to \(\mathcal{A}\) for every \(c \in \mathbb{R}\). In this proposition, the sets \(\{f(x) \leq c\}\), \(\{f(x) > c\}\), or \(\{f(x) \geq c\}\), could be used equally well.

**Theorem 12.24:** If \((f_n)\) is a sequence of measurable functions that converges pointwise to a function \(f\), then \(f\) is measurable. If \((X, \mathcal{A}, \mu)\) is a complete measure space and \((f_n)\) converges pointwise-a.e. to \(f\), then \(f\) is measurable.

**Definition 12.25:** A function \(\varphi : X \to \mathbb{R}\) on a measurable space \((X, \mathcal{A})\) is a **simple function** if there are measurable sets \(A_1, A_2, \ldots, A_n\) and real numbers \(c_1, c_2, \ldots, c_n\) such that

\[ \varphi = \sum_{i=1}^{n} c_i \chi_{A_i}. \]

Here, \(\chi_A\) is the characteristic function of the set \(A\), meaning that

\[ \chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \]

**Theorem 12.26:** Let \(f : X \to [0, \infty]\) be a nonnegative, measurable function. There is a monotone increasing sequence \(\{\varphi_n\}\) of simple function that converges pointwise to \(f\).

12.3 Integration

**Definition 12.27:** Let \(f : X \to [0, \infty]\) be a nonnegative measurable function on a measure space \((X, \mathcal{A}, \mu)\). We define

\[ \int f \, d\mu = \sup \left\{ \int \varphi \, d\mu \mid \varphi \text{ is simple and } \varphi \leq f \right\}. \]

if \(f : X \to \mathbb{R}\) and \(f = f^+ - f^-\), where \(f^+\) and \(f^-\) are the positive and negative parts of \(f\), the we define

\[ \int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu, \]

provided that at least one of the integrals on the right had side is finite. If

\[ \int |f| \, d\mu = \int f^+ \, d\mu + \int f^- \, d\mu < \infty, \]

then we say that \(f\) is **integrable** or **summable**. The Lebesgue integral does not assign a value to the integral of a highly oscillatory function \(f\) for which both \(\int f^+ \, d\mu\) and \(\int f^- \, d\mu\) are infinite.

**Example 12.29:** If \(\delta_{x_0}\) is the delta-measure, and \(f : \mathbb{R}^n \to \mathbb{R}\) is a Borel measurable function, then

\[ \int f \, d\delta_{x_0} = f(x_0). \]

We have \(f = g\) a.e. with respect to \(\delta_{x_0}\) if and only if \(f(x_0) = g(x_0)\).
Example 12.30: Let \( \nu \) be the counting measure on the set \( \mathbb{N} \) of natural numbers defined in Example 12.6. If \( f : \mathbb{N} \to \mathbb{R} \), then
\[
\int f \, d\nu = \sum_{n=1}^{\infty} f_n,
\]
where \( f_n = f(n) \).

### 12.4 Convergence Theorems

**Theorem 12.33 (Monotone Convergence):** Suppose that \((f_n)\) is monotone increasing sequence of non-negative, measurable functions \( f_n : X \to [0, \infty] \) on a measurable space \( (X, \mathcal{A}, \mu) \). Let \( f : X \to [0, \infty] \) be the pointwise limit,
\[
f(x) = \lim_{n \to \infty} f_n(x).
\]
Then
\[
\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu.
\]

**Theorem 12.34 (Fatou):** If \((f_n)\) is any sequence of nonnegative measurable functions \( f_n : X \to [0, \infty] \) on a measure space \( (X, \mathcal{A}, \mu) \), then
\[
\int \left( \liminf_{n \to \infty} f_n \right) \, d\mu \leq \liminf_{n \to \infty} \int f_n \, d\mu.
\]

**Theorem 12.35 (Lebesgue Dominated Convergence):** Suppose that \((f_n)\) is a sequence of integrable functions, \( f_n : X \to \mathbb{R} \) on a measure space \( (X, \mathcal{A}, \mu) \) that converge to a pointwise limiting function \( f : X \to \mathbb{R} \). If there is an integrable function \( g : X \to [0, \infty] \) such that
\[
|f_n(x)| \leq g(x) \quad \text{for all } x \in X \text{ and } n \in \mathbb{N},
\]
then \( f \) is integrable and
\[
\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu.
\]

### 12.5 Product Measure and Fubini’s Theorem

**Definition 12.38:** Let \((X, \mathcal{A})\) and \((Y, \mathcal{B})\) be measurable spaces. The product \(\sigma\)-algebra \( \mathcal{A} \otimes \mathcal{B} \) is the \(\sigma\)-algebra on \( X \times Y \) that is generated by the collection of sets
\[
\{ A \times B \mid A \in \mathcal{A}, B \in \mathcal{B} \}
\]

**Theorem 12.39:** Suppose that \((X, \mathcal{A}, \mu)\) and \((Y, \mathcal{B}, \nu)\) are \(\sigma\)-finite measure spaces. There is a unique product measure \( \mu \otimes \nu \), defined on \( \mathcal{A} \otimes \mathcal{B} \), with the property that for every \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \)
\[
(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B).
\]

**Theorem 12.41 (Fubini):** Let \((X, \mathcal{A}, \mu)\) and \((Y, \mathcal{B}, \nu)\) are \(\sigma\)-finite measure spaces. Suppose that \( f : X \times Y \to \mathbb{R} \) is an \( (\mathcal{A} \otimes \mathcal{B}) \)-measurable function.
(a) The function $f$ is integrable, meaning that
\[ \int_{X \times Y} |f| d\mu \otimes d\nu < \infty, \]
if and only if either of the following iterated integrals is finite:
\[ \int_X \left( \int_Y |f_x(y)| d\nu(y) \right) d\mu(x), \]
\[ \int_Y \left( \int_X |f_y(x)| d\mu(x) \right) d\nu(y). \]

(b) If $f$ is integrable, then
\[ \int_{X \times Y} f(x, y) d(\mu(x) \otimes \nu(y)) = \int_X \left( \int_Y f_x(y) d\nu(y) \right) d\mu(x), \]
\[ \int_{X \times Y} f(x, y) d(\mu(x) \otimes \nu(y)) = \int_Y \left( \int_X f_y(x) d\mu(x) \right) d\nu(y). \]

12.6 The $L^p$ Spaces

**Definition 12.45:** Let $(X, \mathcal{A}, \mu)$ be a measure space and $1 \leq p < \infty$. The space $L^p(X, \mu)$ is the space of equivalence classes of measurable functions $f : X \to \mathbb{C}$, with respect to the equivalence relation of a.e.-equality, such that
\[ \int |f|^p d\mu < \infty. \]
The $L^p$-norm of $f$ is defined by
\[ \|f\|_p = \left( \int_X |f|^p d\mu \right)^{1/p}. \]

**Theorem 12.46:** If $(X, \mathcal{A}, \mu)$ is a measure space and $1 \leq p \leq \infty$, then $L^p(X)$ is a Banach space.

**Theorem 12.48:** Suppose that $(X, \mathcal{A}, \mu)$ is a measure space and $1 \leq p \leq \infty$. If $f \in L^p(X)$, then there is a sequence $(\varphi_n)$ of simple functions $\varphi_n : X \to \mathbb{C}$ such that
\[ \lim_{n \to \infty} \|f - \varphi_n\|_p = 0. \]

**Theorem 12.49:** If $1 \leq p < \infty$, then $L^p(\mathbb{R}^n)$ is a separable metric space.

**Theorem 12.50:** The space $C_c(\mathbb{R}^n)$ of continuous functions with compact support is dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

**Theorem 12.51:** If $1 \leq p < \infty$, then $C_c^\infty(\mathbb{R}^n)$ is a dense subspace of $L^p(\mathbb{R}^n)$. 

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12.7 The Basic Inequalities

**Theorem 12.54 (Hölder):** Let $1 \leq p, p' \leq \infty$ satisfy $1/p + 1/p' = 1$. If $f \in L^p(X, \mu)$ and $g \in L^{p'}(X, \mu)$, then $fg \in L^1(X, \mu)$ and
\[
\left| \int f g d\mu \right| \leq \|f\|_p \|g\|_{p'}.
\]

**Proposition 12.55:** Suppose that $(X, \mu)$ is a finite measure space, meaning that $\mu(X) < \infty$, and $1 \leq q \leq p \leq \infty$. Then
\[
L^1(X, \mu) \supset L^q(X, \mu) \supset L^p(X, \mu) \supset L^\infty(X, \mu).
\]

**Theorem 12.56 (Minkowski):** If $1 \leq p \leq \infty$, and $f, g \in L^p(X, \mu)$, then $f + g \in L^p(X, \mu)$, and
\[
\|f + g\|_p \leq \|f\|_p + \|g\|_p.
\]

**Theorem 12.58 (Young):** Suppose that $1 \leq p, q, r \leq \infty$, satisfy
\[
\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r},
\]
if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g \in L^r(\mathbb{R}^n)$, and
\[
\|f * g\|_r \leq \|f\|_p \|g\|_q.
\]

12.8 The Dual Space of $L^p$

**Theorem 12.59:** If $1 < p < \infty$, then every $\varphi \in L^p(X)^*$ is of the form
\[
\varphi(f) = \int_X fg \, d\mu
\]
for some $g \in L^{p'}(X)$, where $1/p + 1/p' = 1$. If $\mu$ is $\sigma$-finite the same conclusion holds when $p = 1$ and $p' = \infty$. Moreover,
\[
\|\varphi\|_{L^p} = \|g\|_{L^{p'}}.
\]

**Definition 12.60:** Suppose that $1 \leq p < \infty$. A sequence $(f_n)$ converges weakly to $f$ in $L^p$, written $f_n \rightharpoonup f$, if
\[
\lim_{n \to \infty} \int f_n g \, d\mu = \int f g \, d\mu \quad \text{for every } g \in L^{p'},
\]
where $p'$ is the Hölder conjugate of $p$. When $p = \infty$ and $p' = 1$, the condition above corresponds to weak*-convergence in $L^\infty$.

**Theorem 12.62:** Suppose that $(f_n)$ is a bounded sequence in $L^p(X)$, meaning that there is a constant $M$ such that $\|f_n\| \leq M$ for every $n \in \mathbb{N}$. If $1 < p < \infty$, then there is a subsequence $(f_{n_k})$ and a function $f \in L^p(X)$ with $\|f\| \leq M$ such that $f_{n_k} \rightharpoonup f$ as $k \to \infty$ weakly in $L^p(X)$. 

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Other Useful Things to Remember

**Proposition 5.30:** Let $T : X \to Y$ be a bounded linear map between Banach spaces $X,Y$. The following statements are equivalent:

(a) There is a constant $c > 0$ such that

$$c\|x\| \leq \|Tx\| \quad \text{for all } x \in X;$$

(b) $T$ has closed range, and the only solution of the equation $Tx = 0$ is $x = 0$

**Proposition 5.43:** Let $X$, $Y$, $Z$ be Banach spaces.

(a) If $S,T \in B(X,Y)$ are compact, then any linear combination of $S$ and $T$ is compact.

(b) If $(T_n)$ is a sequence of compact operator in $B(X,Y)$ converging uniformly to $T$, then $T$ is compact.

(c) If $T \in B(X,Y)$ has finite-dimensional range, then $T$ is compact.

(d) Let $S \in B(X,Y)$, $T \in B(Y,Z)$. If $S$ is bounded and $T$ is compact, or $S$ is compact and $T$ is bounded, then $TS \in B(X,Z)$ is compact.