ADI - Alternating Direction Implicit

Invented 1955 - Peaceman, Rachford, Douglas
Wide class of methods; illustrated here for \( \frac{du}{dt} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \).

Each version by itself
1st order accurate,
Stability condition
\[ \frac{B}{h^2} \leq \text{const} \]
Basic idea:
Turn 90° between adjacent time levels.

Equations:
\[
\begin{align*}
v(x,y,t+\frac{k}{2}) - v(x,y,t) &= \frac{1}{2} \delta t \left[ (D_x + D_y) v(x,y,t+\frac{k}{2}) + (D_x - D_y) v(x,y,t) \right] \\
v(x,y,t+k) - v(x,y,t+\frac{k}{2}) &= \frac{1}{2} \delta t \left[ (D_x - D_y) v(x,y,t+\frac{k}{2}) + (D_x + D_y) v(x,y,t+k) \right]
\end{align*}
\]

Notation, above:
\[
(D_x) v \begin{cases} \begin{align*} x+y, & x-\text{direction} \rightarrow \text{Forward} \end{align*} \end{cases}
\]

So, for:
\[
(D_x D_y) v(x,y,t) = \frac{1}{h^2} (v(x+h,y,t) - 2v(x,y,t) + v(x-h,y,t)).
\]
Stability analysis

von Neumann: \( v(x,y,t) = e^{iωx} e^{iγy} \)

Each state has its own \( γ \), so consider \( γ_1, γ_2 \) and \( γ = γ_1, γ_2 \).

Simplify notation: \( w_1 h = s_1, w_2 h = s_2 \), \( α = \frac{v_2}{v_1} \).

We then get:

\[
\begin{bmatrix} 1 + α(1 - ω s_1) \end{bmatrix} \begin{bmatrix} γ_1 \\ γ_2 \end{bmatrix} = \begin{bmatrix} 1 - α(1 - ω s_2) \\ 1 + α(1 - ω s_2) \end{bmatrix} \begin{bmatrix} γ_1 \\ γ_2 \end{bmatrix}.
\]

i.e.

\[
γ = \frac{1 - α(1 - ω s_1)}{1 + α(1 - ω s_2)}, \quad \frac{1 - α(1 - ω s_2)}{1 + α(1 - ω s_1)}
\]

\[
= \frac{1 - α(1 - ω s_1)}{1 + α(1 - ω s_2)}.
\]

\[|γ| \leq 1 \Rightarrow \text{Unconditional stability}.
\]

Accuracy analysis

(Check)

Key idea: Eliminate second level:

\[ E_2(1) + E_2(2) \Rightarrow (3) = (1) - (2) \]

\[ E_1(1) + E_1(2) \Rightarrow (4) = (1) + (2) \]

We can use the explicit \( t + \frac{h}{2} \) entry in (3) to eliminate all the \( t + \frac{h}{2} \) entries in (4):

Then get:
Compare this against the known second order Crank-Nicholson (CN) stencil. Difference turns out to be $O(h^2) + O(k^2)$, so method still second order.

Big benefit over CN:
Just need to solve tridiagonal linear system.