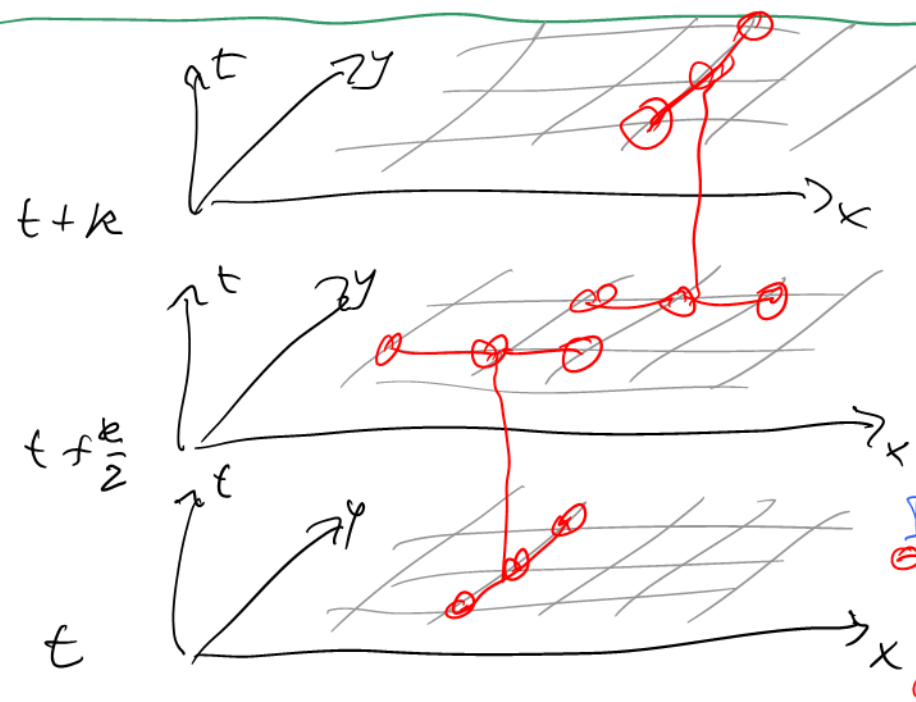
ADI – Alternating Direction Implicit

Invented 1955 – Peaceman, Rachford, Douglas

Wide class of methods; illustrated here for $\frac{\partial u}{\partial t} = \sigma\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$.



Each version by itself 1st order accurate, Stability condition

$k/h^2 \le$ const.

Basic idea:

Turn 90° between adjacent time levels.

Equations:

$$v(x,y,t+\tfrac{k}{2}) - v(x,y,t) = \tfrac{1}{2}\sigma k\left[(D_+D_-)_x v(x,y,t+\tfrac{k}{2}) + (D_+D_-)_y v(x,y,t)\right] \qquad (1)$$

$$v(x,y,t+k) - v(x,y,t+\tfrac{k}{2}) = \tfrac{1}{2}\sigma k\left[(D_+D_-)_x v(x,y,t+\tfrac{k}{2}) + (D_+D_-)_y v(x,y,t+k)\right] \qquad (2)$$

Notation above:

$$(D_+)_x v(x,y,t) = \frac{1}{h}\left(v(x+h,y,t) - v(x,y,t)\right)$$

In x-direction

Forward.

So, for ex.

$$(D_+D_-)_x v(x,y,t) = \frac{1}{h^2}\left(v(x-h,t) - 2v(x,y,t) + v(x+h,y,t)\right).$$

Stability analysis

von Neumann: $v(x,y,t) = \eta^{t/k} e^{i\omega_1 x} \cdot e^{i\omega_2 y}$

Each stage has its own η, so consider η_1, η_2 and $\eta = \eta_1 \cdot \eta_2$.

Simplify notation: $\omega_1 h = s_1$, $\omega_2 h = s_2$, $\alpha = \frac{\sigma k}{h^2}$.

We then get

$$[1+\alpha(1-\cos s_1)]\,\eta_1 = [1-\alpha(1-\cos s_2)]$$
$$[1+\alpha(1-\cos s_2)]\,\eta_2 = [1-\alpha(1-\cos s_1)]$$

i.e.

$$\eta = \underbrace{\frac{1-\alpha(1-\cos s_1)}{1+\alpha(1-\cos s_2)}}_{\eta_1} \cdot \underbrace{\frac{1-\alpha(1-\cos s_2)}{1+\alpha(1-\cos s_1)}}_{\eta_2} =$$

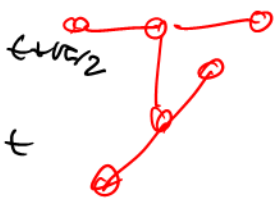
$$= \frac{1-\alpha(1-\cos s_1)}{1+\alpha(1-\cos s_1)} \cdot \frac{1-\alpha(1-\cos s_2)}{1+\alpha(1-\cos s_2)} \Rightarrow |\eta| \le 1$$

$\Rightarrow$ Unconditional stability

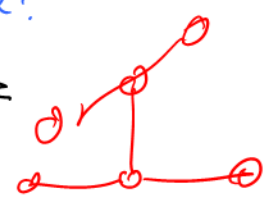
Accuracy analysis (sketch)

Key idea: Eliminate second level:

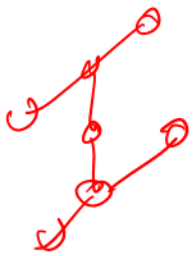
Eq (1)



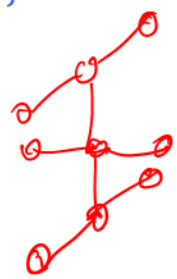
Eq (2)



$\Rightarrow$ (3) = (1) − (2)

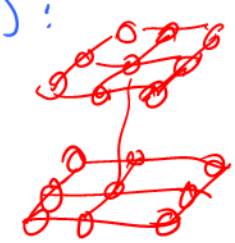


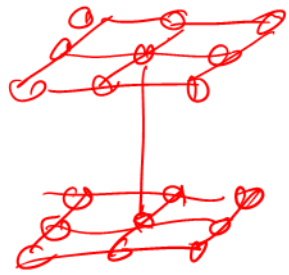
(4) = (1) + (2)



We can use the explicit $t+\frac{k}{2}$ entry in (3) to eliminate all the $t+\frac{k}{2}$ entries in (4):

Then get:



Compare this 

against the known second order Crank-Nicholson (CN) stencil. Difference turns out to be $O(h^2) + O(k^2)$, so method still second order.

Big benefit over CN:
Just need to solve tridiagonal linear systems.
