Absolutely cts fcn, Radon-Nikodym Derivative APPM 5450 Spring 2018 Applied Analysis 2

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Definition 1 (Royden and Fitzpatrick §6.4). *A real-valued function f on a closed, bounded interval $[a,b]$ is said to be **absolutely continuous** on $[a,b]$ provided for each $\epsilon > 0$, there is $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a,b),*

$$\text{if } \sum_{k=1}^{n}(b_k - a_k) < \delta, \text{ then } \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon.$$

If we have an absolutely continuous function f on $[a,b]$, then for every $\epsilon > 0$, there is a $\delta > 0$ such that, in particular using $n = 1$ open intervals $(x, x+\delta)$ (or $(x-\delta, x)$), then $|f(x+\delta) - f(x)| < \epsilon$. That is, not only is f continuous at x, but the choice of δ did not depend on x, so f is in fact uniformly continuous.

Overall, we have absolutely continuous $\implies$ uniformly continuous $\implies$ continuous and, (on a compact interval), Continuously diff. $\implies$ Lipschitz cts $\implies$ absolutely cts $\implies$ bounded variation $\implies$ diff. a. e. where "diff" means differentiable. The Cantor function (p. 32 in Hunter and Nachtergaele) is continuous everywhere but not absolutely continuous. The function $f(x) = \sqrt{x}$ on $[0,1]$ is absolutely continuous but not Lipschitz.

Theorem 2 (**Fund. Thm. Calc.**, Thm. 10 §6.5 Royden and Fitzpatrick, or wikipedia). *If f is absolutely continuous on the finite interval $[a,b]$ then f is differentiable a.e. on (a,b), its derivative f' is integrable, and*

$$\int_a^x f' = f(x) - f(a) \; \forall x \in [a,b].$$

Note that absolute continuity has a different (though related) definition in regards to measures:

Definition 3 (cf. wikipedia). *A measure μ on Borel subsets of the real line $\mathcal{R}(\mathbb{R})$ is **absolutely continuous** with respect to Lebesgue measure λ if for every measurable set A, $\lambda(A) = 0$ implies $\mu(A) = 0$. This is written as $\mu \ll \lambda$ and we say μ is "dominated" by λ.*

Theorem 4. *Let μ be a finite measure on $\mathcal{R}(\mathbb{R})$, then $\mu \ll \lambda$ iff there exists a Lebesgue integrable function g on the real line such that*

$$\forall A \in \mathcal{R}(\mathbb{R}), \quad \mu(A) = \int_A g \, d\lambda. \tag{1}$$

*The function g is unique up to a set of zero measure (wrt λ), and is called the **Radon-Nikodym** derivative of μ, and is often denoted $g = \frac{d\mu}{d\lambda}$.*

The theorem generalizes to $\mathbb{R}^n$, and to general σ-finite measure spaces. The theorem states that a probability measure has a pdf iff it is an absolutely continuous measure. It also implies the following intuitive statement that if $\int_A g \, d\lambda = 0$ for all A, then $g = 0$ a.e. (by uniqueness of the R-N derivative).

In particular, that if a CDF is absolutely continuous, then its PDF is a valid function, not a distribution. For example, discrete measures can not write their PDF as a function (rather, it is a series of delta functions).

A weaker version would just say that if the CDF is differentiable, then it has a PDF (and the PDF is the derivative of the CDF). But that's too stringent. For example, consider a PDF on the interval $[-1,1]$ where the PDF is 1/3 on $[-1,0]$ and 2/3 on $[0,1]$. This is a valid probability distribution, but the CDF is not differentiable at zero. The CDF is absolutely continuous though.

Theorem 5 (Tonelli). *Let $f : X \times Y \to [0,\infty]$ be **non-negative** and measurable and the measures on X and Y be σ-finite, then*

$$\int_X \left(\int_Y f(x,y) dy \right) dx = \int_Y \left(\int_X f(x,y) dx \right) dy = \int_{X \times Y} f(x,y) dy dx$$

1. If $g : [0, \infty) \to \mathbb{R}$ is a monotone non-increasing (thus measurable) function satisfying $\lim_{x\to\infty} g(x) = c > 0$, prove that there exists a rational-valued function $h : [0, \infty) \to \mathbb{Q}$ such that the function $f \colon [0, \infty) \to \mathbb{R}$ defined by $f = g \cdot h$ is improperly Riemann integrable on $[0, \infty)$, but not Lebesgue integrable there.

2. Assume that $f \colon [1, 2] \to \mathbb{R}$ is absolutely continuous, with $f(2) = 0$. Prove that

$$\left| \int_1^2 f'(x) \log x \, \mathrm{d}x \right| \leq \int_1^2 |f(x)| \, \mathrm{d}x.$$

3. Let $f : [a, b] \to \mathbb{R}$ be a C^1 function. For $\epsilon > 0$, let $C_\epsilon := \{x \in (a, b) : |f'(x)| < \epsilon\}$, and let $A := \{f(x) \mid x \in (a, b), f'(x) = 0\}$.

(i) Prove that C_ϵ is open and that $m(f(C_\epsilon)) < \epsilon \cdot (b - a)$.

(ii) Prove that A has Lebesgue measure zero.

4. Let $(X, \mathcal{B}, \mu)$ be a measure space, and suppose that $p, q, r \in (1, \infty)$ satisfy

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1.$$

If $f \in L^p(X, \mu)$, $g \in L^q(X, \mu)$, and $h \in L^r(X, \mu)$, prove that $f \cdot g \cdot h \in L^1(X, \mu)$ and that

$$\|f \cdot g \cdot h\|_1 \leq \|f\|_p \cdot \|g\|_q \cdot \|h\|_r.$$

5. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space, and suppose that $f : X \to [0, \infty)$ is a nonnegative integrable function. Prove that the function $\psi : [0, \infty) \to [0, \infty]$ defined by $\psi(t) = \mu(\{x \in X : f(x) \geq t\})$ is Lebesgue measurable and that

$$\int_X f d\mu = \int_0^\infty \psi(t) \, dt.$$

Hint: you may find Tonelli's Theorem useful.

6. If $\{f_1, f_2, \cdots\}$ is a complete orthonormal set in the Hilbert space $L^2[0, 1]$, where $[0, 1]$ is equipped with Lebesgue measure, and B is an arbitrary measurable subset of positive measure in $[0, 1]$, use Parseval's identity applied to the characteristic function for B to prove that

$$1 \leq \int_B \sum_{i=1}^\infty |f_i(x)|^2 dx.$$