Absolutely continuous functions, Radon-Nikodym Derivative APPM 5450 Spring 2016 Applied Analysis 2

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Definition 1 (Royden and Fitzpatrick §6.4)**.** *A real-valued function f on a closed, bounded interval $[a, b]$ is said to be* ***absolutely continuous*** *on $[a, b]$ provided for each $\epsilon > 0$, there is $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b),*

$$\text{if } \sum_{k=1}^{n}(b_k - a_k) < \delta, \text{ then } \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon.$$

If we have an absolutely continuous function f on $[a, b]$, then for every $\epsilon > 0$, there is a $\delta > 0$ such that, in particular using $n = 1$ open intervals $(x, x + \delta)$ (or $(x - \delta, x)$), then $|f(x + \delta) - f(x)| < \epsilon$. That is, not only is f continuous at x, but the choice of δ did not depend on x, so f is in fact uniformly continuous.

Overall, we have

$$\text{absolutely continuous} \implies \text{uniformly continuous} \implies \text{continuous}$$

and (on a compact interval)

$$\text{Continuously differentiable} \implies \text{Lipschitz cts} \implies \text{absolutely cts} \implies \text{bounded variation} \implies \text{differentiable a. e.}$$

The Cantor function (p. 32 in Hunter and Nachtergaele) is continuous everywhere but not absolutely continuous. The function $f(x) = \sqrt{x}$ on $[0, 1]$ is absolutely continuous but not Lipschitz.

Theorem 2 (**Fund. Thm. Calc.**, Thm. 10 §6.5 Royden and Fitzpatrick, or wikipedia)**.** *If f is absolutely continuous on the finite interval $[a, b]$ then f is differentiable a.e. on (a, b), its derivative f' is integrable, and*

$$\int_a^x f' = f(x) - f(a) \; \forall x \in [a, b].$$

Note that absolute continuity has a different (though related) definition in regards to measures:

Definition 3 (cf. wikipedia)**.** *A measure μ on Borel subsets of the real line $\mathcal{R}(\mathbb{R})$ is* ***absolutely continuous*** *with respect to Lebesgue measure λ if for every measurable set A, $\lambda(A) = 0$ implies $\mu(A) = 0$. This is written as $\mu \ll \lambda$ and we say μ is "dominated" by λ.*

Theorem 4. *Let μ be a finite measure on $\mathcal{R}(\mathbb{R})$, then $\mu \ll \lambda$ iff there exists a Lebesgue integrable function g on the real line such that*

$$\forall A \in \mathcal{R}(\mathbb{R}), \quad \mu(A) = \int_A g \, d\lambda. \tag{1}$$

The function g is unique up to a set of zero measure (wrt λ), and is called the ***Radon-Nikodym*** *derivative of μ, and is often denoted $g = \frac{d\mu}{d\lambda}$.*

The theorem generalizes to $\mathbb{R}^n$, and to general σ-finite measure spaces. The theorem states that a probability measure has a pdf iff it is an absolutely continuous measure. It also implies the following intuitive statement that if $\int_A g \, d\lambda = 0$ for all A, then $g = 0$ a.e. (by uniqueness of the R-N derivative).

Theorem 5 (Tonelli)**.** *Let $f : X \times Y \to [0, \infty]$ be* ***non-negative*** *and measurable and the measures on X and Y be σ-finite, then*

$$\int_X \left(\int_Y f(x, y) dy \right) dx = \int_Y \left(\int_X f(x, y) dx \right) dy = \int_{X \times Y} f(x, y) dy dx$$

1. If $g : [0, \infty) \to \mathbb{R}$ is a monotone non-increasing (thus measurable) function satisfying $\lim_{x\to\infty} g(x) = c > 0$, prove that there exists a rational-valued function $h : [0, \infty) \to \mathbb{Q}$ such that the function $f \colon [0, \infty) \to \mathbb{R}$ defined by $f = g \cdot h$ is improperly Riemann integrable on $[0, \infty)$, but not Lebesgue integrable there.

2. Assume that $f \colon [1, 2] \to \mathbb{R}$ is absolutely continuous, with $f(2) = 0$. Prove that

$$\left| \int_1^2 f'(x) \log x \, \mathrm{d}x \right| \;\le\; \int_1^2 |f(x)| \, \mathrm{d}x.$$

3. Let $f : [a, b] \to \mathbb{R}$ be a C^1 function. For $\epsilon > 0$, let $C_\epsilon := \{x \in (a, b) : |f'(x)| < \epsilon\}$, and let $A := \{f(x) \mid x \in (a, b), f'(x) = 0\}$.

(i) Prove that C_ϵ is open and that $m(f(C_\epsilon)) < \epsilon \cdot (b - a)$.

(ii) Prove that A has Lebesgue measure zero.

4. Let $(X, \mathcal{B}, \mu)$ be a measure space, and suppose that $p, q,\ r \;\in\; (1, \infty)$ satisfy

$$\frac{1}{p} \;+\; \frac{1}{q} \;+\; \frac{1}{r} \;=\; 1.$$

If $f \;\in\; L^p(X, \mu),\ g \;\in\; L^q(X, \mu)$, and $h \;\in\; L^r(X, \mu)$, prove that $f \cdot g \cdot h \;\in\; L^1(X, \mu)$ and that

$$\|f \cdot g \cdot h\|_1 \;\le\; \|f\|_p \cdot \|g\|_q \cdot \|h\|_r.$$

5. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space, and suppose that $f : X \to [0, \infty)$ is a nonnegative integrable function. Prove that the function $\psi : [0, \infty) \to [0, \infty]$ defined by $\psi(t) = \mu(\{x \in X : \ f(x) \ge t\})$ is Lebesgue measurable and that

$$\int_X f d\mu \;=\; \int_0^\infty \;\psi(t)\; dt.$$

Hint: you may find Tonelli's Theorem useful.

6. If $\{f_1, f_2, \cdots\}$ is a complete orthonormal set in the Hilbert space $L^2[0, 1]$, where $[0, 1]$ is equipped with Lebesgue measure, and B is an arbitrary measurable subset of positive measure in $[0, 1]$, use Parseval's identity applied to the characteristic function for B to prove that

$$1 \;\le\; \int_B \sum_{i=1}^\infty |f_i(x)|^2 dx.$$