Absolutely continuous functions, Radon-Nikodym Derivative
APPM 5450 Spring 2016 Applied Analysis 2

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**Definition 1** (Royden and Fitzpatrick §6.4). A real-valued function \( f \) on a closed, bounded interval \([a, b]\) is said to be **absolutely continuous** on \([a, b]\) if for each \( \epsilon > 0 \), there is \( \delta > 0 \) such that for every finite disjoint collection \( \{(a_k, b_k)\}_{k=1}^{n} \) of open intervals in \((a, b)\),

\[
\text{if } \sum_{k=1}^{n}(b_k - a_k) < \delta, \text{ then } \sum_{k=1}^{n}|f(b_k) - f(a_k)| < \epsilon.
\]

If we have an absolutely continuous function \( f \) on \([a, b] \), then for every \( \epsilon > 0 \), there is a \( \delta > 0 \) such that, in particular using \( n = 1 \) open intervals \((x, x + \delta)\) (or \((x - \delta, x)\)), then \( |f(x + \delta) - f(x)| < \epsilon \). That is, not only is \( f \) continuous at \( x \), but the choice of \( \delta \) did not depend on \( x \), so \( f \) is in fact uniformly continuous.

Overall, we have

\[
\text{absolutely continuous } \implies \text{uniformly continuous } \implies \text{continuous}
\]

and (on a compact interval)
Continuously differentiable \( \implies \) Lipschitz cts \( \implies \) absolutely cts \( \implies \) bounded variation \( \implies \) differentiable a.e.

The Cantor function (p. 32 in Hunter and Nachtergaele) is continuous everywhere but not absolutely continuous. The function \( f(x) = \sqrt{x} \) on \([0, 1]\) is absolutely continuous but not Lipschitz.

**Theorem 2 (Fund. Thm. Calc., Thm. 10 §6.5 Royden and Fitzpatrick, or wikipedia)**. If \( f \) is absolutely continuous on the finite interval \([a, b]\) then \( f \) is differentiable a.e. on \((a, b)\), its derivative \( f' \) is integrable, and

\[
\int_{a}^{x} f' = f(x) - f(a) \quad \forall x \in [a, b].
\]

Note that absolute continuity has a different (though related) definition in regards to measures:

**Definition 3** (cf. wikipedia). A measure \( \mu \) on Borel subsets of the real line \( \mathcal{B}(\mathbb{R}) \) is **absolutely continuous** with respect to Lebesgue measure \( \lambda \) if for every measurable set \( A \), \( \lambda(A) = 0 \) implies \( \mu(A) = 0 \). This is written as \( \mu \ll \lambda \) and we say \( \mu \) is “dominated” by \( \lambda \).

**Theorem 4.** Let \( \mu \) be a finite measure on \( \mathcal{B}(\mathbb{R}) \), then \( \mu \ll \lambda \) iff there exists a Lebesgue integrable function \( g \) on the real line such that

\[
\forall A \in \mathcal{B}(\mathbb{R}), \quad \mu(A) = \int_{A} g \, d\lambda.
\]  \hspace{1cm} (1)

The function \( g \) is unique up to a set of zero measure (wrt \( \lambda \)), and is called the **Radon-Nikodym derivative** of \( \mu \), and is often denoted \( g = \frac{d\mu}{d\lambda} \).

The theorem generalizes to \( \mathbb{R}^n \), and to general \( \sigma \)-finite measure spaces. The theorem states that a probability measure has a pdf iff it is an absolutely continuous measure. It also implies the following intuitive statement that if \( \int_A g \, d\lambda = 0 \) for all \( A \), then \( g = 0 \) a.e. (by uniqueness of the R-N derivative).

**Theorem 5** (Tonelli). Let \( f : X \times Y \to [0, \infty] \) be **non-negative** and measurable and the measures on \( X \) and \( Y \) be \( \sigma \)-finite, then

\[
\int_{X} \left( \int_{Y} f(x, y) \, dy \right) \, dx = \int_{Y} \left( \int_{X} f(x, y) \, dx \right) \, dy = \int_{X \times Y} f(x, y) \, dy \, dx
\]
1. If \( g : [0, \infty) \to \mathbb{R} \) is a monotone non-increasing (thus measurable) function satisfying
\[
\lim_{x \to \infty} g(x) = c > 0,
\]
prove that there exists a rational-valued function \( h : [0, \infty) \to \mathbb{Q} \) such that the function \( f : [0, \infty) \to \mathbb{R} \) defined by \( f = g \cdot h \) is improperly Riemann integrable on \([0, \infty)\), but not Lebesgue integrable there.

2. Assume that \( f : [1, 2] \to \mathbb{R} \) is absolutely continuous, with \( f(2) = 0 \). Prove that
\[
\int_1^2 f'(x) \log x \, dx \leq \int_1^2 |f(x)| \, dx.
\]

3. Let \( f : [a, b] \to \mathbb{R} \) be a \( C^1 \) function. For \( \epsilon > 0 \), let \( C_\epsilon := \{ x \in (a, b) : |f'(x)| < \epsilon \} \), and let \( A := \{ f(x) \mid x \in (a, b), f'(x) = 0 \} \).

(i) Prove that \( C_\epsilon \) is open and that \( m(f(C_\epsilon)) < \epsilon \cdot (b - a) \).

(ii) Prove that \( A \) has Lebesgue measure zero.

4. Let \((X, B, \mu)\) be a measure space, and suppose that \( p, q, r \in (1, \infty) \) satisfy
\[
\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1.
\]
If \( f \in L^p(X, \mu), \ g \in L^q(X, \mu), \) and \( h \in L^r(X, \mu) \), prove that \( f \cdot g \cdot h \in L^1(X, \mu) \) and that
\[
\|f \cdot g \cdot h\|_1 \leq \|f\|_p \cdot \|g\|_q \cdot \|h\|_r.
\]

5. Let \((X, B, \mu)\) be a \( \sigma \)-finite measure space, and suppose that \( f : X \to [0, \infty) \) is a nonnegative integrable function. Prove that the function \( \psi : [0, \infty) \to [0, \infty] \) defined by \( \psi(t) = \mu(\{ x \in X : f(x) \geq t \}) \) is Lebesgue measurable and that
\[
\int_X f \, d\mu = \int_0^\infty \psi(t) \, dt.
\]

**Hint:** you may find Tonelli’s Theorem useful.

6. If \( \{ f_1, f_2, \cdots \} \) is a complete orthonormal set in the Hilbert space \( L^2[0, 1] \), where \([0, 1]\) is equipped with Lebesgue measure, and \( B \) is an arbitrary measurable subset of positive measure in \([0, 1]\), use Parseval’s identity applied to the characteristic function for \( B \) to prove that
\[
1 \leq \int_B \sum_{i=1}^\infty |f_i(x)|^2 \, dx.
\]