Problem 1: Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series.

(a) Prove that if $a_n \geq 0$ for all $n$, then
\[
\sum_{n=1}^{\infty} a_n^2
\]
converges.

(b) By contrast, find an example of a sequence $\{a_n\}$ for which the series converges, but (1) diverges.

(c) Suppose that $\{b_n\}$ is a bounded sequence, and that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Prove that
\[
\sum_{n=1}^{\infty} a_n b_n
\]
converges.

Problem 2: Consider the following two sequences of functions:
\[
f_k(x) = \begin{cases} 
1, & x \in [0, \frac{1}{k}] \\
3(\frac{1}{k} - x) + 1, & x \in [\frac{1}{k}, \frac{1}{k} + \frac{1}{3}] \\
0, & x \in [\frac{1}{k} + \frac{1}{3}, 1]
\end{cases}
\] and
\[
g_k(x) = \begin{cases} 
1, & x \in [0, \frac{1}{k}] \\
3(1 - kx) + 1, & x \in [\frac{1}{k}, \frac{3}{4k}] \\
0, & x \in [\frac{3}{4k}, 1]
\end{cases}
\]
for $k \geq 2$, $k$ an integer. To which sequences does the Arzelà-Ascoli theorem apply and why? What does the theorem allow one to conclude?

Problem 3: Suppose that $f$ is integrable on $\mathbb{R}^d$. Prove that for every $\epsilon > 0$ the following hold:

(a) There exists a set $B$ of finite measure such that
\[
\int_{B} |f| < \epsilon.
\]

(b) There exists a $\delta > 0$ such that
\[
\int_{E} |f| < \epsilon
\]
if the measure of $E$ is less than $\delta$.

Problem 4: Let $T : H \to H$ be a non-trivial, compact and self-adjoint operator on a Hilbert Space $H$. Show that either $-\|T\|$ or $\|T\|$ is an eigenvalue of $T$.

Problem 5: Prove that a closed linear subspace $Y$ of a reflexive Banach space $X$ is also reflexive. (HINT: You might want to use the following result: A point $z$ in a normed vector space $X$ belongs to the closed linear span of a subset $\{y_i\} \subset X$ if and only if for every $\ell \in X^*$ that vanishes on the subset $\{y_i\}$ also vanishes on $z$. That is, if $\ell(y_i) = 0$ for all $y_i$ then $\ell(z) = 0$).