Applied Analysis Preliminary Exam

10.00am–1.00pm, January 10, 2019

Instructions: You have three hours to complete this exam. Work all five problems; each is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are being asked to prove such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name.

Problem 1: Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series.

(a) Prove that if $a_n \geq 0$ for all n, then

$$\sum_{n=1}^{\infty} a_n^2 \tag{1}$$

converges.

(b) By contrast, find an example of a sequence $\{a_n\}$ for which the series converges, but (1) diverges.

(c) Suppose that $\{b_n\}$ is a bounded sequence, and that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Prove that

$$\sum_{n=1}^{\infty} a_n b_n$$

converges.

Problem 2: Consider the following two sequences of functions:

$$f_k(x) = \begin{cases} 1, & x \in [0, \frac{1}{k}] \\ 3(\frac{1}{k} - x) + 1, & x \in [\frac{1}{k}, \frac{1}{k} + \frac{1}{3}] \\ 0, & x \in [\frac{1}{k} + \frac{1}{3}, 1] \end{cases} \quad \text{and} \quad g_k(x) = \begin{cases} 1, & x \in [0, \frac{1}{k}] \\ 3(1 - kx) + 1, & x \in [\frac{1}{k}, \frac{4}{3k}] \\ 0, & x \in [\frac{4}{3k}, 1] \end{cases}$$

for $k \geq 2$, k an integer. To which sequences does the Arzelà-Ascoli theorem apply and why? What does the theorem allow one to conclude?

Problem 3: Suppose that f is integrable on $\mathbb{R}^d$. Prove that for every $\epsilon > 0$ the following hold:

(a) There exists a set B of finite measure such that

$$\int_B |f| < \epsilon.$$

(b) There exists a $\delta > 0$ such that

$$\int_E |f| < \epsilon$$

if the measure of E is less than δ.

Problem 4: Let $T : H \to H$ be a non-trivial, compact and self-adjoint operator on a Hilbert Space H. Show that either $-\|T\|$ or $\|T\|$ is an eigenvalue of T.

Problem 5: Prove that a closed linear subspace Y of a reflexive Banach space X is also reflexive. (HINT: You might want to use the following result: A point z in a normed vector space X belongs to the closed linear span of a subset $\{y_i\} \subset X$ if and only if for every $\ell \in X^*$ that vanishes on the subset $\{y_i\}$ also vanishes on z. That is, if $\ell(y_i) = 0$ for all y_i then $\ell(z) = 0$).