Applied Analysis Preliminary Exam (Hints/solutions)
9.00am–12.00pm, January 13, 2021

Instructions: You have three hours to complete this exam. Work all five problems; each is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are being asked to prove such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name.

Problem 1: Cauchy Condensation Test and an application:

(a) Prove the Cauchy Condensation Test: Suppose a_n is a decreasing sequence with $a_n \geq 0$. Then, $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if and only if $\displaystyle\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.

(b) For what values of a and b will the series $\displaystyle\sum_{n=2}^{\infty} n^{-a}(\log n)^{-b}$ converge?

Solution/Hint:

(a) Consider the partial sums:

$$\begin{aligned} s_n &= \sum_{i=1}^{n} a_i \\ t_k &= \sum_{j=1}^{k} 2^j a_{2^j} \end{aligned}$$

As each a_n is nonnegative, the partial sums s_n and t_k are monotone increasing. Moreover, as $\{a_n\}$ is a decreasing and positive sequence we have that:

For $n < 2^k$,

$$\begin{aligned} s_n &\leq a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + \cdots + 2^k a_{2^k} \\ &= t_k \end{aligned}$$

Thus, $s_n \leq t_k$ for $n < 2^k$.

On the other hand, if $n \geq 2^k$,

$$\begin{aligned} s_n &\geq a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ &\geq (1/2)a_1 + a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k} \\ &= (1/2)t_k \end{aligned}$$

Thus, $2s_n \geq t_k$ when $n \geq 2^k$.

We see that $\{s_n\}$ and $\{t_n\}$ are either both bounded or both unbounded. The Monotone Convergence Theorem for Sequences, which states that a monotone sequence converges if and only if it is bounded, concludes the proof.

(b) Apply the Cauchy Condensation Test to $\displaystyle\sum_{n=2}^{\infty} n^{-a}(\log n)^{-b}$, to obtain the series

$$\sum_{n=2}^{\infty} 2^n (2^n)^{-a} (\log 2^n)^{-b} = \sum_{n=2}^{\infty} (2^n)^{1-a} n^{-b} (\log 2)^{-b}.$$

Apply the ratio test to this series to obtain that it converges when $a > 1$ and diverges when $a < 1$, regardless of the value of b. When $a = 1$ the ratio test is inconclusive.

To handle the case when $a = 1$: Substitute in $a = 1$ to obtain the series $\sum_{n=2}^{\infty} n^{-b}(\log 2)^{-b}$ which converges when $b > 1$ and diverges when $b \leq 1$.

Problem 2: Do the following problems:

(a) State the Arzela-Ascoli Theorem.

(b) Let $C_0(\mathbb{R})$ denote the Banach space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \to 0$ as $|x| \to \infty$, equipped with the sup-norm. A family of functions, $F \subset C_0(\mathbb{R})$, is said to be tight if for all $f \in F$ and every $\epsilon > 0$ there exists $R > 0$ such that $|f(x)| \leq \epsilon$ for all $x \in \mathbb{R}$ with $|x| \geq R$. Prove that $F \subset C_0(\mathbb{R})$ is precompact in $C_0(\mathbb{R})$ if it is bounded, equicontinuous, and tight.

Solution/Hint:

(a) The family of functions $F \subset C([a, b])$ is compact if and only if it is bounded, closed, and equicontinuous.

(b) Give that $C_0(\mathbb{R})$ is a Banach Space then it is complete. Thus, F is precompact if and only if it is totally bounded. Now, we show that if F is bounded, equicontinuous, and tight then it is totally bounded. For this purpose, let $\epsilon > 0$. Since F is tight, there exists an $R > 0$ such that for all f it holds that:

$$|f(x)| < \frac{\epsilon}{2} \quad \text{for} \quad |x| \geq R.$$

Define $f_R : [-R, R] \to \mathbb{R}$ to be the restriction of f to $[-R, R]$. By assumption we know that family $\{f_R : f \in F\}$ is bounded and equicontinuous in the space $C([-R, R])$. Thus, by the Arzelá-Ascoli theorem it is precompact, which implies that it is totally bounded. Thus, $\{f_R : f \in F\}$ has a finite-ϵ net: $\{f_R^1, f_R^2, \ldots, f_R^n\} \subset \{f_R : f \in F\}$ for some $f^k \in F$. Now, for $f \in F$ we have that for $|x| \geq R$

$$\left|f(x) - f^i(x)\right| \leq |f| + \left|f^i\right| < \epsilon.$$

Thus, $f_R \in B_\epsilon(f_R^i)$ implies that $f \in B_\epsilon(f^i)$. From this, we conclude that $\{f^1, f^2, \ldots, f^n\}$ is a finite-ϵ net of F, which means that its it totally bounded.

Problem 3: Let H be a Hilbert space and $U = \{u_n\}_{n \in \mathbb{N}} \subset H$ an orthonormal set. Consider the map $P_U : H \to H$ defined by:

$$P_U(x) = \sum_{n=1}^{\infty} (u_n, x) u_n.$$

(a) Show that P is a bounded linear operator.

(b) Find the norm of P.

(c) Under what conditions is the operator P_U the identity? Explain.

(d) Show that $P_U^2 = P_U$.

Solution/Hint:

(a) By orthogonality we see that

$$\|P_U(x)\|^2 = \left\|\sum_{n=1}^{\infty} (u_n, x) u_n\right\|^2 = \sum_{n=1}^{\infty} \|(u_n, x) u_n\|^2 = \sum_{n=1}^{\infty} |(u_n, x)|^2 \leq \|x\|,$$

where the last inequality follows from Bessel's inequality. Thus, P_U is bounded. Linearity follows by the linearity of the second argument of the inner-product.

(b) From part (a) we know that $\|P\| \leq 1$. We also see that for all $k \in \mathbb{N}$ we have, by orthogonality, that

$$P_U(u_k) = \sum_{n=1}^{\infty} (u_n, u_k) u_n = (u_k, u_k) u_k = u_k.$$

Thus, we have that

$$\|P_U(u_k)\| = \|u_k\|$$

so $\|P_U\| = 1$.

(c) We need the orthonormal set U to be complete, in other words, if it has to be a a basis. In this case, we have that

$$P_U(x) = x,$$

and so it is the identity.

(d) Here we have that:

$$\begin{aligned} P_U^2(x) &= P_U\left(\sum_{n=1}^{\infty}(u_n, x)u_n\right) \\ &= \sum_{n=1}^{\infty}(u_n, x)P_U(u_n) \quad \text{by linearity of } P_U \\ &= \sum_{n=1}^{\infty}(u_n, x)u_n \\ &= P_U(x). \end{aligned}$$

Problem 4: Solve the following unrelated problems:

(a) Show that the set of polynomials with domain $[0,1]$ is dense in $(C^1([0,1]), \|\cdot\|_{C^1})$.

(b) Let X be a vector space. Show that if $P : X \to X$ is a projection, then $X = ran(P) \oplus ker(P)$.

(c) Let H be a finite dimensional Hilbert Space and suppose that $T : H \to H$ is self-adjoint. Moreover, suppose that for $\lambda \in \mathbb{R}$ and $\epsilon > 0$ there is an $x \in H$ with unit norm such that

$$\|Tx - \lambda x\| \leq \epsilon.$$

Show that T has an eigenvalue μ such that $|\lambda - \mu| < \epsilon$.

Solution/Hint:

(a) Let $f \in C^1([0,1])$ then $f' \in C([0,1])$. Let $\epsilon > 0$, by the Weierstrass Approximation Theorem there is a polynomial defined on $[0,1]$, q, such that

$$\left\|f' - q\right\|_\infty < \frac{\epsilon}{2}.$$

Define

$$p(x) = \int_0^x q(s)\, ds + f(0).$$

We see that p is a polynomial and that $p' = q$. Note that we can write

$$f(x) = \int_0^x f'(s)\, ds + f(0),$$

so we can compute:

$$|f(x) - p(x)| = \left|\int_0^x [f'(s) - q(s)]\, ds\right| \leq \int_0^x \left|f'(s) - q(s)\right|\, ds \leq x\left\|f' - q\right\|_\infty \leq \frac{\epsilon}{2}.$$

Now, taking the sup over $[0,1]$ gives that $\|f - p\|_\infty < \frac{\epsilon}{2}$. Then, we have that $\|f - p\|_{C^1} < \epsilon$ and from this we conclude.

(b) This is Theorem 8.2 (a) in our textbook, but for completeness we reproduce the proof here. The first step is to prove that $x \in ran(P)$ iff $Px = x$. If $Px = x$ then $x \in ran(P)$. Now, assume that $x \in ran(P)$, so there exists a $y \in X$ such that $Py = x$. Now, P is a projections, which by definition is a linear map such that $P^2 = P$, so we have that $Px = P^2y = Py = x$. If $x \in ran(P) \cap ker(P)$ then based on the above argument we see that $Px = x$ and $Px = 0$, which means that $x = 0$. Thus, $ran(P) \cap ker(P) = \{0\}$. Now, if $x \in H$ then we have that

$$x = Px + (x - Px)$$

where $Px \in ran(x)$ and as $P(x - Px) = Px - P^2x = Px - Px = 0$ then $x - Px \in ker(P)$. This was true for any arbitrary $x \in X$ so we see that $X = ran(P) \oplus ker(P)$.

(c) Since T is self-adjoint, so is $T - \lambda I$. Thus, by the spectral theorem it is orthonormally diagonalizable: let $e_1, e_2, \ldots, e_n$ be an orthonormal basis with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. We can express $x = \sum_{k=1}^n \alpha_k e_k$ and we have

$$\begin{aligned}
\|(T-\lambda I)x\|^2 &= \left\| (T-\lambda I) \sum_{k=1}^n \alpha_k e_k \right\|^2 \\
&= \left\| \sum_{k=1}^n \lambda_k \alpha_k e_k \right\|^2 \\
&= \sum_{k=1}^n |\lambda_k \alpha_k|^2 \\
&= \sum_{k=1}^n \left| \min_{0 \le j \le n} \lambda_j \right|^2 |\alpha_k|^2 \\
&\ge \left| \min_{0 \le j \le n} \lambda_j \right|^2 .
\end{aligned}$$

Thus, we have if $\lambda_m = \min_{0 \le j \le n} \lambda_j$ then

$$|\lambda_m| \le \|(T - \lambda I)x\| < \epsilon.$$

However, note that $\mu = \lambda_m + \lambda$ is an eigenvalue of T, so $|\mu - \lambda| = |\lambda_m| \le \epsilon$.

Problem 5: Show that for every non-negative, bounded, and measurable function $f : [0,1] \to \mathbb{R}$, it holds that:

$$\int_0^1 f(x)\, d\mu(x) = \inf \int_0^1 \varphi(x)\, d\mu(x)$$

where the infimum is taken over all simple measurable functions φ that satisfy $f \le \varphi$. Here μ is the Lebesgue measure.

Solution/Hint: Let $M = \sup_{x\in[0,1]} f$, which is finite since f is bounded. For every simple function $\varphi \ge f$, we can define another simple function $\psi(x) = \min\{\varphi, M\}$. Note that $f \le \psi \le \varphi$. Of course, we have that

$$\int_0^1 \psi(x)\, d\mu(x) \le \int_0^1 \varphi(x)\, d\mu(x)$$

and so

$$\inf_{f \le \psi \le M} \int_0^1 \psi\, d\mu(x) \le \inf_{f \le \varphi} \int_0^1 \varphi\, d\mu(x).$$

This means that we can restrict our search for the infimum over the set of simple functions that satisfy $f \le \psi \le M$. Now, note that $g = M - f$ is a non-negative, measurable function that is bounded by M. Thus, by definition of the integral

$$M - \int_0^1 f\, d\mu(x) = \int_0^1 g\, d\mu(x) = \sup_{0 \le \phi \le g} \int_0^1 \phi\, d\mu(x),$$

ϕ a simple function. Note that each ϕ defines a simple function

$$\psi = M - \phi$$

that satisfies $f \le \psi \le M$. At the same time, for all simple function ψ satisfying $f \le \psi \le M$ we have a simple function $\phi = \psi - M$ that satisfies $0 \le \phi \le g$. Thus, we obtain that:

$$\int_0^1 f\, d\mu(x) = M - \sup_{0 \le \phi \le g} \int_0^1 \phi\, d\mu(x) = - \sup_{f \le \psi \le M} \int_0^1 (-\psi)\, d\mu(x) = \inf_{f \le \psi \le M} \int_0^1 \psi\, d\mu(x).$$