Problem 1: Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series.

(a) Prove that if $a_n \geq 0$ for all $n$, then

$$\sum_{n=1}^{\infty} a_n^2$$

converges.

(b) By contrast, find an example of a sequence $\{a_n\}$ for which the series converges, but (1) diverges.

(c) Suppose that $\{b_n\}$ is a bounded sequence, and that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Prove that

$$\sum_{n=1}^{\infty} a_n b_n$$

converges.

Solution/Hint:

(a) Note that when $a_n \geq 0$, then the partial sum

$$\sum_{n=1}^{M} a_n^2 \leq \left( \sum_{n=1}^{M} a_n \right)^2,$$

since the added cross terms are all non-negative. Since the sequence of partial sums $s_M = \sum_{n=1}^{M} a_n \to s$ converges, then we have

$$\sum_{n=1}^{M} a_n^2 \leq (s_M)^2 \leq s$$

Now $\sum_{n=1}^{M} a_n^2$ is a monotone increasing, bounded sequence, and so it has a limit.

(b) Let, for example $a_n = (-1)^n/\sqrt{n}$. This alternating series converges by the alternating series test but since $a_n^2 = 1/n$ is the Harmonic series, it diverges.

(c) Since the sequence is assumed to be absolutely convergent then its absolute values give a Cauchy sequence: for all $\epsilon > 0$ there exists an $N(\epsilon)$ such that whenever $M > N$

$$\sum_{n=N}^{M} |a_n| < \epsilon$$

Thus

$$\left| \sum_{n=N}^{M} a_n b_n \right| \leq \sum_{n=N}^{M} |a_n||b_n| \leq B \sum_{n=N}^{M} |a_n| < B\epsilon$$

where by assumption $|b_n| \leq B < \infty$. Therefore the sequence $s_k = \sum_{n=1}^{k} a_n b_n$ is a Cauchy sequence: For all $\epsilon' = B\epsilon > 0$

$$|s_M - s_N| = \left| \sum_{n=N}^{M} a_n b_n \right| < \epsilon'$$

with $M > N$ as before. Thus $s_k$ converges.
**Problem 2:** Consider the following two sequences of functions:

\[ f_k(x) = \begin{cases} 
1, & x \in [0, \frac{1}{k}] \\
3\left(\frac{1}{k} - x\right) + 1, & x \in \left[\frac{1}{k}, \frac{1}{k} + \frac{1}{3}\right] \\
0, & x \in \left[\frac{1}{k} + \frac{1}{3}, 1\right]
\end{cases} \quad \text{and} \quad g_k(x) = \begin{cases} 
1, & x \in [0, \frac{1}{k}] \\
3(1 - kx) + 1, & x \in \left[\frac{1}{k}, \frac{4}{3k}\right] \\
0, & x \in \left[\frac{4}{3k}, 1\right]
\end{cases} \]

for \( k \geq 2, k \) an integer. To which sequences does the Arzelà-Ascoli theorem apply and why? What does the theorem allow one to conclude?

**Solution/Hint:** Both sequences are continuous functions on \([0, 1]\) for all \( k \), and hence uniformly bounded. The first sequence is equicontinuous since

\[ |f_k(x) - f_k(y)| \leq 3|x - y| < \epsilon \]

if \( |x - y| < \delta \) and we choose \( \delta = \epsilon/3 \) independently of \( k \). Thus the theorem implies this sequence has a uniformly convergent subsequence. Indeed the sequence itself converges uniformly to a continuous function, so every subsequence also converges to the same function. The second sequence, however is not equicontinuous, for if \( x, y \) are in the middle interval then

\[ |g_k(x) - g_k(y)| = 3k|x - y| \]

so it would be necessary to choose \( \delta = \epsilon/(3k) \), which depends on \( k \). Thus the theorem does not apply. Indeed, this sequence converges pointwise to the discontinuous function \( g(x) = 1 \) if \( x = 0 \) and \( 0 \) otherwise.

**Problem 3:** Suppose that \( f \) is integrable on \( \mathbb{R}^d \). Prove that for every \( \epsilon > 0 \) the following hold:

(a) There exists a set \( B \) of finite measure such that

\[ \int_B |f| < \epsilon. \]

(b) There exists a \( \delta > 0 \) such that

\[ \int_E |f| < \epsilon \]

if the measure of \( E \) is less than \( \delta \).

**Solution/Hint:**

- WLOG assume that \( f \geq 0 \). To prove (a) let \( B_N \) be the ball of radius \( N \) centered at 0 and define

\[ f_N(x) = f(x)\chi_{B_N}(x), \]

where \( \chi \) is the characteristic function. Note that \( f_N \geq 0 \) and is measurable. Moreover, \( f_N(x) \leq f_{N+1}(x) \) and

\[ \lim_{N \to \infty} f_N(x) = f(x). \]

- By the Monotone Convergence Theorem we have that

\[ \lim_{N \to \infty} \int f_N = \int f. \]

Hence, for \( \epsilon > 0 \) there exists a \( N \) sufficiently large such that

\[ 0 \leq \int f - \int f_N < \epsilon. \]

- However, note that \( \chi_{B_N} = 1 - \chi_{B_N} \). Thus, (2) implies that \( \int_{B_N^C} f < \epsilon. \)

- To prove part (b) we now define \( f_N(x) = f(x)\chi_{E_N} \) where

\[ E_N = \{x: f(x) \leq N\}. \]

- Note that \( f_N \geq 0 \) is measurable and again \( f_N(x) \leq f_{N+1}(x) \).

- Again for any \( \epsilon > 0 \) by the Monotone Convergence Theorem there exists an \( N \) such that

\[ \int (f - f_N) < \epsilon/2. \]
Choose $\delta > 0$ such that $N\delta < \epsilon/2$. If the measure of $E$ is less than $\delta$ (that is $m(E) < \delta$) then we have that

$$
\int_E f = \int_E (f - f_N) + \int_E f_N \\
\leq \int (f - f_N) + \int f_N \\
\leq \epsilon/2 + Nm(E) \\
\leq \epsilon.
$$

**Problem 4:** Let $T : H \to H$ be a non-trivial, compact and self-adjoint operator on a Hilbert Space $H$. Show that either $-\|T\|$ or $\|T\|$ is an eigenvalue of $T$.

**Solution/Hint:**

- Let $m = \|T\| = \sup_{\|f\|=1} |(Tf, f)|$ by Lemma 8.26 in H-N book. Thus either,

  $$
  \|T\| = \sup_{\|f\|=1} (Tf, f) \quad \text{or} \quad -\|T\| = \inf_{\|f\|=1} (Tf, f).
  $$

- First assume that the former holds (the proof for the latter is similar).
- Take a sequence $\{f_n\} \subset H$ with $\|f_n\| = 1$ and $(Tf_n, f_n) \to m$.

- Since $T$ is compact there exists a converging subsequence (not renamed for convenience) $Tf_n \to g$ in $H$.
- **Claim:** $g$ is an eigenvector of $T$ with eigenvalue $m$.
- Indeed, note that

  $$
  \|Tf_n - mf_n\|^2 = \|Tf_n\|^2 - 2m(Tf_n, f_n) + m^2\|f_n\|^2 \\
  \leq \|T\|^2\|f_n\|^2 - 2m(Tf_n, f_n) + m^2\|f_n\|^2 \\
  = m^2 - 2m(Tf_n, f_n) \to 0
  $$

  as $n \to \infty$. Thus, $Tf_n - mf_n \to 0$ as $n \to \infty$. Thus, $Tf_n \to g$ implies that $mf_n \to g$.
- Now, as $g$ is continuous we have that $mTf_n \to Tg$.
- Finally, note $g$ is non-trivial for it were then $\|Tf_n\| \to 0$ and so $(Tf_n, f_n) \to 0$ and then $\|T\| = 0$ which is a contradiction.

**Problem 5:** Prove that a closed linear subspace $Y$ of a reflexive Banach space $X$ is also reflexive. (HINT: You might want to use the following result: A point $z$ in a normed vector space $X$ belongs to the closed linear span of a subset $\{y_i\} \subset X$ if and only if for every $\ell \in X^*$ that vanishes on the subset $\{y_i\}$ also vanishes on $z$. That is, if $\ell(y_i) = 0$ for all $y_i$ then $\ell(z) = 0$).

**Solution/Hint:**

- Let $\ell \in X^*$ then $\ell|_Y$ is a bounded linear functional on $Y$.
- Denote $\ell_0 = \ell|_Y : Y \to F$ (where $F$ denotes the field).
- By Hahn-Banach all bounded, linear functional can be extended to $X$. Thus, the restriction $R : X^* \to Y^*$ defined by

  $$
  \ell R \ell_0
  $$

  is onto.
- This restriction induces a mapping from $Y^{**} \to X^{**}$ as follows: for any $\eta \in Y^{**}$ we define $\gamma \in X^{**}$ by setting

  $$
  \gamma(\ell) = \eta(\ell_0) \quad \forall \ell \in X^*.
  $$

- $X$ is reflexive and so $\gamma$ can be identified with an element $z \in X$: $\gamma(\ell) = \ell(z)$ and thus

  $$(3) \quad \ell(z) = \eta(\ell_0).$$
• **Claim:** $z \in Y$. Indeed, if $\ell \in Y^\perp$ so that $\ell(y) = 0$ for all $y \in Y$ then $\ell_0 = 0$ and thus by (3) then $\ell(z) = 0$.

• By the hint then $z$ is in the closure of $Y$. However, as $Y$ is closed then $z \in Y$.

• Thus, we can write (3) as

$$\ell_0(z) = \eta_0.$$  \hfill (4)

• However, every functional in $Y^*$ occurs as $\ell_0$ so (4) shows that every $\eta \in Y^{**}$ can be identified with a $z \in Y$. Thus, $Y$ is reflexive.