Applied Analysis Preliminary Exam (Hints/solutions)
10.00am–1.00pm, January 10, 2019

Instructions: You have three hours to complete this exam. Work all five problems; each is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are being asked to prove such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name.

Problem 1: Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series.

(a) Prove that if $a_n \geq 0$ for all n, then

$$\sum_{n=1}^{\infty} a_n^2 \tag{1}$$

converges.

(b) By contrast, find an example of a sequence $\{a_n\}$ for which the series converges, but (1) diverges.

(c) Suppose that $\{b_n\}$ is a bounded sequence, and that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Prove that

$$\sum_{n=1}^{\infty} a_n b_n$$

converges.

Solution/Hint:

(a) Note that when $a_n \geq 0$, then the partial sum

$$\sum_{n=1}^{M} a_n^2 \leq \left(\sum_{n=1}^{M} a_n\right)^2,$$

since the added cross terms are all non-negative. Since the sequence of partial sums $s_M = \sum_{n=1}^{M} a_n \to s$ converges, then we have

$$\sum_{n=1}^{M} a_n^2 \leq (s_M)^2 \leq s$$

Now $\sum_{n=1}^{M} a_n^2$ is a monotone increasing, bounded sequence, and so it has a limit.

(b) Let, for example $a_n = (-1)^n/\sqrt{n}$. This alternating series converges by the alternating series test but since $a_n^2 = 1/n$ is the Harmonic series, it diverges.

(c) Since the sequence is assumed to be absolutely convergent then its absolute values give a Cauchy sequence: for all $\epsilon > 0$ there exists an $N(\epsilon)$ such that whenever $M > N$

$$\sum_{n=N}^{M} |a_n| < \epsilon$$

Thus

$$\left|\sum_{n=N}^{M} a_n b_n\right| \leq \sum_{n=N}^{M} |a_n||b_n| \leq B \sum_{n=N}^{M} |a_n| < B\epsilon$$

where by assumption $|b_n| \leq B < \infty$. Therefore the sequence $s_k = \sum_{n=1}^{k} a_n b_n$ is a Cauchy sequence: For all $\epsilon' = B\epsilon > 0$

$$|s_M - s_N| = \left|\sum_{n=N}^{M} a_n b_n\right| < \epsilon'$$

with $M > N$ as before. Thus s_k converges.

Problem 2: Consider the following two sequences of functions:

$$f_k(x) = \begin{cases} 1, & x \in [0, \frac{1}{k}] \\ 3(\frac{1}{k} - x) + 1, & x \in [\frac{1}{k}, \frac{1}{k} + \frac{1}{3}] \\ 0, & x \in [\frac{1}{k} + \frac{1}{3}, 1] \end{cases} \quad \text{and} \quad g_k(x) = \begin{cases} 1, & x \in [0, \frac{1}{k}] \\ 3(1 - kx) + 1, & x \in [\frac{1}{k}, \frac{4}{3k}] \\ 0, & x \in [\frac{4}{3k}, 1] \end{cases}$$

for $k \geq 2$, k an integer. To which sequences does the Arzelà-Ascoli theorem apply and why? What does the theorem allow one to conclude?

Solution/Hint: Both sequences are continuous functions on $[0, 1]$ for all k, and hence uniformly bounded. The first sequence is equicontinuous since

$$|f_k(x) - f_k(y)| \leq 3|x - y| < \epsilon$$

if $|x - y| < \delta$ and we choose $\delta = \epsilon/3$ independently of k. Thus the theorem implies this sequence has a uniformly convergent subsequence. Indeed the sequence itself converges uniformly to a continuous function, so every subsequence also converges to the same function. The second sequence, however is not equicontinuous, for if x, y are in the middle interval then

$$|g_k(x) - g_k(y)| = 3k|x - y|$$

so it would be necessary to choose $\delta = \epsilon/(3k)$, which depends on k. Thus the theorem does not apply. Indeed, this sequence converges pointwise to the discontinuous function $g(x) = 1$ if $x = 0$ and 0 otherwise.

Problem 3: Suppose that f is integrable on $\mathbb{R}^d$. Prove that for every $\epsilon > 0$ the following hold:

(a) There exists a set B of finite measure such that

$$\int_B |f| < \epsilon.$$

(b) There exists a $\delta > 0$ such that

$$\int_E |f| < \epsilon$$

if the measure of E is less than δ.

Solution/Hint:

- WLOG assume that $f \geq 0$. To prove (a) let B_N be the ball of radius N centered at 0 and define

$$f_N(x) = f(x)\chi_{B_N}(x),$$

where χ is the characteristic function. Note that $f_N \geq 0$ and is measurable. Moreover, $f_N(x) \leq f_{N+1}(x)$ and

$$\lim_{N\to\infty} f_N(x) = f(x).$$

- By the Monotone Convergence Theorem we have that

$$\lim_{N\to\infty} \int f_N = \int f.$$

Hence, for $\epsilon > 0$ there exists a N sufficiently large such that

$$0 \leq \int f - \int f_N < \epsilon. \tag{2}$$

- However, note that $\chi_{B_N^c} = 1 - \chi_{B_N}$. Thus, (2) implies that $\int_{B_N^c} f < \epsilon$.
- To prove part (b) we now define $f_N(x) = f(x)\chi_{E_N}$ where

$$E_N = \{x : f(x) \leq N\}.$$

- Note that $f_N \geq 0$ is measurable and again $f_N(x) \leq f_{N+1}(x)$.
- Again for any $\epsilon > 0$ by the Monotone Convergence Theorem there exists an N such that

$$\int (f - f_N) < \epsilon/2.$$

Choose $\delta > 0$ such that $N\delta < \epsilon/2$. If the measure of E is less than δ (that is $m(E) < \delta$) then we have that

$$\begin{aligned}\int_E f &= \int_E (f - f_N) + \int_E f_N \\ &\le \int (f - f_N) + \int_E f_N \\ &\le \epsilon/2 + Nm(E) \\ &\le \epsilon.\end{aligned}$$

Problem 4: Let $T : H \to H$ be a non-trivial, compact and self-adjoint operator on a Hilbert Space H. Show that either $-\|T\|$ or $\|T\|$ is an eigenvalue of T.

Solution/Hint:

- Let $m = \|T\| = \sup_{\|f\|=1} |(Tf, f)|$ by Lemma 8.26 in H-N book. Thus either,
$$\|T\| = \sup_{\|f\|=1} (Tf, f) \quad \text{or} \quad -\|T\| = \inf_{\|f\|=1} (Tf, f).$$
- First assume that the former holds (the proof for the latter is similar).
- Take a sequence $\{f_n\} \subset H$ with $\|f_n\| = 1$ and $(Tf_n, f_n) \to m$.

- Since T is compact there exists a converging subsequence (not renamed for convenience) $Tf_n \to g$ in H.
- *Claim:* g is an eigenvector of T with eigenvalue m.
- Indeed, note that
$$\begin{aligned}\|Tf_n - mf_n\|^2 &= \|Tf_n\|^2 - 2m(Tf_n, f_n) + m^2\|f_n\|^2 \\ &\le \|T\|^2\|f_n\|^2 - 2m(Tf_n, f_n) + m^2\|f_n\|^2 \\ &= 2m^2 - 2m(Tf_n, f_n) \to 0\end{aligned}$$
as $n \to \infty$. Thus, $Tf_n - mf_n \to 0$ as $n \to \infty$. Thus, $Tf_n \to g$ implies that $mf_n \to g$.
- Now, as g is continuous we have that $mTf_n \to Tg$.
- Finally, note g is non-trivial for it were then $\|Tf_n\| \to 0$ and so $(Tf_n, f_n) \to 0$ and then $\|T\| = 0$ which is a contradiction.

Problem 5: Prove that a closed linear subspace Y of a reflexive Banach space X is also reflexive. (HINT: You might want to use the following result: A point z in a normed vector space X belongs to the closed linear span of a subset $\{y_i\} \subset X$ if and only if for every $\ell \in X^*$ that vanishes on the subset $\{y_i\}$ also vanishes on z. That is, if $\ell(y_i) = 0$ for all y_i then $\ell(z) = 0$).

Solution/Hint:

- Let $\ell \in X^*$ then $\ell|_Y$ is a bounded linear functional on Y.
- Denote $\ell_0 = \ell|_Y : Y \to \mathbb{F}$ (where $\mathbb{F}$ denotes the field).
- By Hahn-Banach all bounded, linear functional can be extended to X. Thus, the restriction $R : X^* \to Y^*$ defined by
$$\ell \xrightarrow{R} \ell_0$$
is onto.
- This restriction induces a mapping from $Y^{**} \to X^{**}$ as follows: for any $\eta \in Y^{**}$ we define $\gamma \in X^{**}$ by setting
$$\gamma(\ell) = \eta(\ell_0) \quad \forall\, \ell \in X^*.$$
- X is reflexive and so γ can be identified with an element $z \in X$: $\gamma(\ell) = \ell(z)$ and thus
$$\ell(z) = \eta(\ell_0). \tag{3}$$

- **Claim:** $z \in Y$. Indeed, if $\ell \in Y^{\perp}$ so that $\ell(y) = 0$ for all $y \in Y$ then $\ell_0 = 0$ and thus by (3) then $\ell(z) = 0$.
- By the hint then z is in the closure of Y. However, as Y is closed then $z \in Y$.
- Thus, we can write (3) as

$$\ell_0(z) = \eta(\ell_0). \tag{4}$$

- However, every functional in Y^* occurs as ℓ_0 so (4) shows that every $\eta \in Y^{**}$ can be identified with a $z \in Y$. Thus, Y is relexive.