Applied Analysis Preliminary Exam

9.00am–12.00pm, January 13, 2021

Instructions: You have three hours to complete this exam. Work all five problems; each is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are being asked to prove such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name.

Problem 1: Cauchy Condensation Test and an application:

(a) Prove the Cauchy Condensation Test: Suppose a_n is a decreasing sequence with $a_n \geq 0$. Then, $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if and only if $\displaystyle\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.

(b) For what values of a and b will the series $\displaystyle\sum_{n=2}^{\infty} n^{-a}(\log n)^{-b}$ converge?

Problem 2: Do the following problems:

(a) State the Arzelá-Ascoli Theorem.

(b) Let $C_0(\mathbb{R})$ denote the Banach space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \to 0$ as $|x| \to \infty$, equipped with the sup-norm. A family of functions, $F \subset C_0(\mathbb{R})$, is said to be tight if for all $f \in F$ and every $\epsilon > 0$ there exists $R > 0$ such that $|f(x)| \leq \epsilon$ for all $x \in \mathbb{R}$ with $|x| \geq R$. Prove that $F \subset C_0(\mathbb{R})$ is precompact in $C_0(\mathbb{R})$ if it is bounded, equicontinuous, and tight.

Problem 3: Let H be a Hilbert space and $U = \{u_n\}_{n\in\mathbb{N}} \subset H$ an orthonormal set. Consider the map $P_U : H \to H$ defined by:

$$P_U(x) = \sum_{n=1}^{\infty} (u_n, x)u_n.$$

(a) Show that P is a bounded linear operator.

(b) Find the norm of P.

(c) Under what conditions is the operator P_U the identity? Explain.

(d) Show that $P_U^2 = P_U$.

Problem 4: Solve the following unrelated problems:

(a) Show that the set of polynomials with domain $[0, 1]$ is dense in $(C^1([0, 1]), \|\cdot\|_{C^1})$.

(b) Let X be a vector space. Show that if $P : X \to X$ is a projection, then $X = ran(P) \oplus ker(P)$.

(c) Let H be a finite dimensional Hilbert Space and suppose that $T : H \to H$ is self-adjoint. Moreover, suppose that for $\lambda \in \mathbb{R}$ and $\epsilon > 0$ there is an $x \in H$ with unit norm such that

$$\|Tx - \lambda x\| \leq \epsilon.$$

Show that T has an eigenvalue μ such that $|\lambda - \mu| < \epsilon$.

Problem 5: Show that for every non-negative, bounded, and measurable function $f : [0, 1] \to \mathbb{R}$, it holds that:

$$\int_0^1 f(x)\, d\mu(x) = \inf \int_0^1 \varphi(x)\, d\mu(x)$$

where the infimum is taken over all simple measurable functions φ that satisfy $f \leq \varphi$. Here μ is the Lebesgue measure.