Applied Analysis Preliminary Exam (Hints/solutions)
10.00am–1.00pm, August 21, 2019

Instructions: You have three hours to complete this exam. Work all five problems; each is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are being asked to prove such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name.

Problem 1: Let $\{x_n\}$, $\{a_n\}$, and $\{b_n\}$ be sequences in $\mathbb{R}$.

(a) Suppose $x_n \to x$ converges and that for each n, $x_n \in [a_n, b_n]$. Show that

$$\limsup_{n\to\infty} a_n \le x \le \liminf_{n\to\infty} b_n.$$

(b) If for each n, $a_n, b_n > 0$ and $b_n \to b > 0$ converges show that

$$\limsup_{n\to\infty} a_n b_n = b \limsup_{n\to\infty} a_n.$$

Are the positivity assumptions necessary for this result?

Solution/Hint:

(a) By the bounds $x_n \le b_n$, so $\inf_{k>n} x_k \le \inf_{k>n} b_k$, and

$$x = \lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n \le \liminf_{n\to\infty} b_n$$

The other side is similarly shown.

(b) Since b_n converges, for any $\epsilon > 0$ there is an $N(\epsilon)$ such that $|b_n - b| < \epsilon$. Thus whenever $n > N$, $a_n(b-\epsilon) < a_n b_n < a_n(b+\epsilon)$, and if $k > N$, and $\epsilon < b$,

$$(b-\epsilon)\sup_{n>k} a_n \le \sup_{n>k} a_n b_n \le (b+\epsilon)\sup_{n>k} a_n$$

so taking limits we have, for any $b > \epsilon > 0$,

$$(b-\epsilon)\limsup_{n\to\infty} a_n \le \limsup_{n>k} a_n b_n \le (b+\epsilon)\limsup_{n\to\infty} a_n$$

which implies the result.

Suppose that $b_n = 1/n$ and $a_n = n$. Then $\limsup a_n b_n = 1$, but $b \limsup a_n = 0 \cdot \infty$ does not exist. So positivity of b is necessary.

Problem 2: Let $g : [0,1] \to \mathbb{R}$ be continuous. Show that there exists a unique continuous function $f : [0,1] \to \mathbb{R}$ such that

$$f(x) - \int_0^x f(x-t)e^{-t^2} dt = g(x).$$

Solution/Hint: Let the operator T be defined by

$$T(f) = g(x) + \int_0^x f(x-t)e^{-t^2} dt$$

Note that if f is continuous then $T(f)$ is as well since g and e^{-t^2} are. Moreover for any continuous functions f and h, and any $x \in [0,1]$

$$\begin{aligned}
||T(f) - T(h)||_\infty &\le \sup_{x\in[0,1]} \int_0^x |f(x-t) - h(x-t)| e^{-t^2} dt \\
&\le \|f-h\|_\infty \sup_{x\in[0,1]} \int_0^x e^{-t^2} dt \\
&\le \|f-h\|_\infty \int_0^1 e^{-t^2} dt \\
&< c\|f-h\|_\infty
\end{aligned}$$

where $c = \int_0^1 e^{-t^2} dt < 1$. Thus T is a contraction, and by the Banach theorem, it has a unique fixed point on the complete space $C([0,1],\mathbb{R})$. Such a fixed point $T(f) = f$ solves the equation.

Problem 3: Consider $H = L^2(\mathcal{S})$ where $\mathcal{S}$ is the unit circle and let $g \in L^1(\mathcal{S})$. Define the operator $K : H \to H$ by

$$K(f) = \int_{\mathcal{S}} g(y)f(x-y)\, dy.$$

(a) Show that K is a bounded operator.
(b) Show that K is a compact operator. (You may use the fact that we can approximate g in L^1 with a sequence of functions $g_n \in L^2$)
(c) Is K a normal operator? Prove or disprove.

Solution/Hint:

(a) This is a consequence of Young's inequality:

$$\|K(f)\|_2 = \|f * g\|_2 \leq \|g\|_1 \|f\|_2.$$

Thus, K is bounded with $\|K\| \leq \|g\|_1$.

(b) We first prove it for $g_n \in L^2$. We proceed by showing that the image of the unit ball, B, in H under K is precompact. For $f \in B$ we use Cauchy-Schwarz to get:

$$\begin{aligned} |Kf(x) - Kf(z)| &= \left| \int_{\mathcal{S}} [g_n(x-y) - g_n(z-y)] f(y)\, dy \right| \\ &\leq \left(\int_{\mathcal{S}} |g_n(x-y) - g_n(z-y)|^2 \, dy \right)^{1/2} \|f\|_2 \\ &= \left(\int_{\mathcal{S}} |g_n(w) - g_n(w+z-x)|^2 \, dw \right)^{1/2} \|f\|_2 \end{aligned}$$

Note that as $x \to z$ the integral above vanishes. Thus, we see that the set of functions in the image of B under K are equi-continous (and uniformly bounded by Youngs inequality $\|K(f)\|_\infty \leq \|g_n\|_2 \|f\|_2$.) and by Arzela-Ascoli this set of functions are pre-compact in the sup-norm. Thus, they are pre-compact in the L^2 norm (weaker norm). Now, let $g_n \in L^2$ be a sequence of functions that converge to g in L^1 and K_n the approximation operators to K (by (a)). Now, noting that the uniform limit of compact operators is compact gives the result.

(c) Consider the L^2 inner-product of $Kf, h \in L^2(\mathcal{S})$:

$$\begin{aligned} (Kf, h) &= \int_{\mathcal{S}} \int_{\mathcal{S}} g(x-y) f(y) \bar{h}(x)\, dydx \\ &= \int_{\mathcal{S}} \int_{\mathcal{S}} g(y-x) \bar{h}(y) f(x)\, dxdy \\ &= (f, K^* g), \end{aligned}$$

where K^* is the convolution operator with function $g^*(x) = \bar{g}(-x)$. Hence, since the adjoint K^* is a convolution it commutes.

Problem 4: Answer the following:

(a) Calculate $\lim_{n\to\infty} n \int_0^1 \sqrt{x} e^{-x^2 n^2}\, dx$.
(b) Calculate $\lim_{n\to\infty} n^2 \int_0^1 x e^{-x^2 n^2}\, dx$. Can you exchange the limit and the integral?
(c) Use Fubini's Theorem and the fact that $\int_{\mathbb{R}} e^{-|x|^2}\, dx = \sqrt{\pi}$. to show that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\, dx = \pi^{n/2}.$$

Solution/Hint:

(a) Let $f_n(x) = nxe^{-x^2 n^2}$ and note that

$$\sup_{x\in[0,1]} f_n(x) \leq \sup_{t>0} te^{-t^2} \leq 1.$$

Therefore,

$$n\sqrt{x}e^{-x^2n^2} \leq \frac{1}{\sqrt{x}}$$

and since

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx < \infty.$$

we can apply DCT and exchange the limit and the integral to get:

$$\lim_{n\to\infty} n\int_0^1 \sqrt{x}e^{-x^2n^2}\,dx = \int_0^1 \lim_{n\to\infty} n\sqrt{x}e^{-x^2n^2}\,dx = \int_0^1 0\,dx = 0.$$

(b) We can compute the integral using substitution:

$$\int_0^1 n^2xe^{-x^2n^2}\,dx = \int_0^1 n^2xe^{-x^2n^2}\,dx = \int_0^{n^2} (1/2)e^{-u}\,du = \frac{1-e^{-n^2}}{2}.$$

Hence,

$$\lim_{n\to\infty} n^2\int_0^1 xe^{-x^2n^2}\,dx = \frac{1}{2}.$$

Moreover, note that

$$\int_0^1 \lim_{n\to\infty} n^2xe^{-x^2n^2}\,dx = \int_0^1 0\,dx = 0$$

and thus we cannot interchange the limit and the integral.

(c) We prove this by induction. The base case is given to us and now we assume that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\,dx = \pi^{n/2}$$

and compute

$$\int_{\mathbb{R}^{n+1}} e^{-|x|^2}\,dx.$$

Let us write $x \in \mathbb{R}^{n+1}$ as $x = (y, x_{n+1})$ where $y \in \mathbb{R}^n$. Let us compute

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}} e^{-|y|^2-|x_{n+1}|^2}\,dx_{n+1}\,dy &= \int_{\mathbb{R}^n} e^{-|y|^2}\int_{\mathbb{R}} e^{-|x_{n+1}|^2}\,dx_{n+1}\,dy\\ &= \sqrt{\pi}\int_{\mathbb{R}^n} e^{-|y|^2}\,dy\\ &= \pi^{\frac{n+1}{2}}.\end{aligned}$$

By Fubini's Theorem the result holds.

Problem 5: Prove that a bounded self-adjoint operator M is non-negative if and only if its spectrum $\sigma(M) \subset [0,\infty)$.

Solution/Hint: ($\Rightarrow$)Note that we know that M is self-adjoint and so $\sigma(M) \subset [-\|M\|, \|M\|]$. Now, assume that $\lambda < 0$ is an eigenvalue let $\mu = -\lambda$. We aim to show that $(M+\mu I)$ is invertible. Indeed, consider that for an arbitrary $x \in H$ we have that

$$\|Mx+\mu x\|^2 = (Mx+\mu x, Mx+\mu x) = \|Mx\|^2 + 2\mu(Mx,x) + \mu^2\|x\|^2 \geq \mu^2\|x\|^2,$$

where we have used that M is non-negative and $\mu > 0$. Thus, by proposition 5.30 in H-N book we know that $(M+\mu I)$ is one-to-one, so μ cannot be in the point spectrum, and $(M+\mu I)$ has closed range so μ cannot be in the continuous spectrum. Remember, we get that the residual spectrum is empty for free (as M is self-adjoint). So it cannot be the spectrum. In this case we have that $\sigma(M) \subset [0, \|M\|]$.

($\Leftarrow$) Since $\sigma(M) \subset [0, \infty)$ we can define the operator $N = \sqrt{M}$ (through functional calculus with $f(\lambda) = \sqrt{\lambda}$ which is continuous for $\lambda > 0$) which is self-adjoint and satisfies $M = N^2$ and we have

$$(Mx, x) = (N^2 x, x) = (Nx, Nx) \geq 0.$$