Problem 1: Let \{x_n\}, \{a_n\}, and \{b_n\} be sequences in \(\mathbb{R}\).

(a) Suppose \(x_n \to x\) converges and that for each \(n\), \(x_n \in [a_n, b_n]\). Show that
\[
\limsup_{n \to \infty} a_n \leq x \leq \liminf_{n \to \infty} b_n.
\]

(b) If for each \(n\), \(a_n, b_n > 0\) and \(b_n \to b > 0\) converges show that
\[
\lim_{n \to \infty} a_n b_n = b \limsup_{n \to \infty} a_n.
\]
Are the positivity assumptions necessary for this result?

Problem 2: Let \(g : [0, 1] \to \mathbb{R}\) be continuous. Show that there exists a unique continuous function \(f : [0, 1] \to \mathbb{R}\) such that
\[
f(x) - \int_0^x f(x-t)e^{-t^2}dt = g(x).
\]

Problem 3: Consider \(H = L^2(S)\) where \(S\) is the unit circle and let \(g \in L^1(S)\). Define the operator \(K : H \to H\) by
\[
K(f) = \int_S g(y)f(x-y)\,dy.
\]

(a) Show that \(K\) is a bounded operator.
(b) Show that \(K\) is a compact operator. (You may use the fact that we can approximate \(g\) in \(L^1\) with a sequence of functions \(g_n \in L^2\)).
(c) Is \(K\) a normal operator? Prove or disprove.

Problem 4: Answer the following:
(a) Calculate \(\lim_{n \to \infty} n \int_0^1 \sqrt{x}e^{-x^2n^2}\,dx\).
(b) Calculate \(\lim_{n \to \infty} n^2 \int_0^1 xe^{-x^2n^2}\,dx\). Can you exchange the limit and the integral?
(c) Use Fubini’s Theorem and the fact that \(\int_\mathbb{R} e^{-|x|^2}\,dx = \sqrt{\pi}\) to show that
\[
\int_{\mathbb{R}^n} e^{-|x|^2}\,dx = \pi^{n/2}.
\]

Problem 5: Prove that a bounded self-adjoint operator \(M\) is non-negative if and only if its spectrum \(\sigma(M) \subset [0, \infty)\).