Applied Analysis Preliminary Exam

10.00am–1.00pm, August 21, 2019

Instructions: You have three hours to complete this exam. Work all five problems; each is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are being asked to prove such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name.

Problem 1: Let $\{x_n\}$, $\{a_n\}$, and $\{b_n\}$ be sequences in $\mathbb{R}$.

(a) Suppose $x_n \to x$ converges and that for each n, $x_n \in [a_n, b_n]$. Show that

$$\limsup_{n\to\infty} a_n \le x \le \liminf_{n\to\infty} b_n.$$

(b) If for each n, $a_n, b_n > 0$ and $b_n \to b > 0$ converges show that

$$\limsup_{n\to\infty} a_n b_n = b \limsup_{n\to\infty} a_n.$$

Are the positivity assumptions necessary for this result?

Problem 2: Let $g : [0,1] \to \mathbb{R}$ be continuous. Show that there exists a unique continuous function $f : [0,1] \to \mathbb{R}$ such that

$$f(x) - \int_0^x f(x-t)e^{-t^2} dt = g(x).$$

Problem 3: Consider $H = L^2(\mathcal{S})$ where $\mathcal{S}$ is the unit circle and let $g \in L^1(\mathcal{S})$. Define the operator $K : H \to H$ by

$$K(f) = \int_{\mathcal{S}} g(y) f(x-y)\, dy.$$

(a) Show that K is a bounded operator.

(b) Show that K is a compact operator. (You may use the fact that we can approximate g in L^1 with a sequence of functions $g_n \in L^2$)

(c) Is K a normal operator? Prove or disprove.

Problem 4: Answer the following:

(a) Calculate $\lim_{n\to\infty} n \int_0^1 \sqrt{x} e^{-x^2 n^2}\, dx$.

(b) Calculate $\lim_{n\to\infty} n^2 \int_0^1 x e^{-x^2 n^2}\, dx$. Can you exchange the limit and the integral?

(c) Use Fubini's Theorem and the fact that $\int_{\mathbb{R}} e^{-|x|^2}\, dx = \sqrt{\pi}$. to show that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\, dx = \pi^{n/2}.$$

Problem 5: Prove that a bounded self-adjoint operator M is non-negative if and only if its spectrum $\sigma(M) \subset [0, \infty)$.