Instructions: You have three hours to complete this exam. Work all five problems; each is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are being asked to prove such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name.

Student Number:

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1. (20 points) Compute the following quantities.

   (a) For each real x, compute $\lim_{n\to\infty} e^{-nx}\left(1+\frac{x}{n}\right)^{n^2}$ (Hint: $\ln(1+x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$ for $-1 < x \leq 1$)

   (b) Evaluate $\lim_{n\to\infty}\left[\frac{1}{\sqrt{n(n+0)}} + \frac{1}{\sqrt{n(n+1)}} + \cdots \frac{1}{\sqrt{n(n+n)}}\right]$

Solutions:

(a) Let

$$\begin{aligned} L &= \lim_{n\to\infty} e^{-nx}\left(1+\frac{x}{n}\right)^{n^2} \\ \ln(L) &= \lim_{n\to\infty}\left[\ln(e^{-nx}) + n^2\ln\left(1+\frac{x}{n}\right)\right] \\ &= \lim_{n\to\infty}\left[-nx + n^2\ln\left(1+\frac{x}{n}\right)\right] \end{aligned}$$

Recall the MacLaurin series for $\ln(1+x)$,

$$\ln(1+x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$$

with an interval of convergence $-1 < x \leq 1$. Then, for fixed positive integer n

$$\ln(1+x/n) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{kn^k}$$

with interval of convergence $-n < x \leq n$ and

$$n^2\ln(1+x/n) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{kn^{k-2}} = nx - \frac{x^2}{2} + \frac{x^3}{3n} - \frac{x^4}{4n^2} + \cdots$$

Substituting this expansion into the expression for $\ln(L)$ gives

$$\begin{aligned}
\ln(L) &= \lim_{n\to\infty}\left[-nx + n^2\ln\left(1+\frac{x}{n}\right)\right] \\
&= \lim_{n\to\infty}\left[-nx + nx - \frac{x^2}{2} + \frac{x^3}{3n} - \frac{x^4}{4n^2} + \cdots\right] \\
&= \lim_{n\to\infty}\left[-\frac{x^2}{2} + \frac{x^3}{3n} - \frac{x^4}{4n^2} + \cdots\right] \\
&= -\frac{x^2}{2}.
\end{aligned}$$

Thus,

$$L = \lim_{n\to\infty} e^{-nx}\left(1+\frac{x}{n}\right)^{n^2} = e^{-x^2/2} \text{ for all } x \in \mathbb{R}.$$

(b) For this limit, we use Reimann sums with n equally spaced subintervals on the interval $[0,1]$ and the Fundamental Theorem of Calculus. Thus:

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{\sqrt{n(n+k)}} &= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{n\sqrt{1+k/n}} \\
&= \lim_{n\to\infty}\left[\sum_{k=0}^{n-1}\frac{1}{n\sqrt{1+k/n}} + \frac{1}{\sqrt{2}n}\right] \\
&= \int_0^1 \frac{1}{\sqrt{1+x}}\,dx \\
&= 2(\sqrt{2}-1).
\end{aligned}$$

2. (20 points) Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction, with contraction constant c. Choose $x_0 \in X$ and define the sequence $\{x_n\}$ by $Tx_n = x_{n+1}$. From the Contraction Mapping Theorem, T has a unique fixed point x. Prove the following inequalities:

   (a) For $n \geq m \geq 1$, $d(x_m, x_n) \leq \dfrac{c^m}{1-c} d(x_1, x_0)$

   (b) $d(x_m, x) \leq \dfrac{c^m}{1-c} d(x_1, x_0)$

   (c) $d(x_m, x) \leq \dfrac{c}{1-c} d(x_{m-1}, x_m)$

Solutions:

(a) Recall a mapping $T : X \to X$ is a contraction if there exists a constant c, with $0 \leq c < 1$ so that $d(T(x), T(y)) \leq cd(x, y)$ for all $x, y \in X$. Since $Tx_n = x_{n+1}$, we have $T^n x_0 = x_n$ So, if $n \geq m \geq 1$ then

$$\begin{aligned}
d(x_n, x_m) &= d(T^n x_0, T^m x_0) \\
&\leq c^m d(T^{n-m} x_0, x_0) \\
&\leq c^m \left[d(T^{n-m} x_0, T^{n-m-1} x_0) + d(T^{n-m-1} x_0, T^{n-m-2} x_0) + \cdots + d(Tx_0, x_0) \right] \\
&\leq c^m \left[\sum_{k=0}^{n-m-1} c^k \right] d(x_1, x_0) \\
&\leq c^m \left[\sum_{k=0}^{\infty} c^k \right] d(x_1, x_0) \\
&\leq \frac{c^m}{1-c} d(x_1, x_0)
\end{aligned}$$

Aside: Since $0 \leq c < 1$ this proves that $\{x_n\}$ is a Cauchy sequence.

(b) By the Contraction Mapping Theorem, we have $\lim_{n\to\infty} x_n = x$. So the inequality in part (b) follows immediately from part (a) since the right-hand side of part (a) is independent of n.

(c) Since x is a fixed point and T is a contraction we have

$$\begin{aligned}
d(x_m, x) &= d(Tx_{m-1}, T(x)) \\
&\leq cd(x_{m-1}, x) \\
&\leq c\left(d(x_{m-1}, x_m) + d(x_m, x)\right) \text{ by the Triangle Inequality)}
\end{aligned}$$

Rearrange the terms to obtain the desired inequality.

Aside: Parts (b) and (c) provide error estimates for the iteration. The inequality in part (b) is sometimes called the prior estimate and the inequality in part (c) is called the posterior estimate.

3. (20 points) Let (X, d) be a compact metric space. Prove that the following set is compact

$$A = \{f \in C(X) : ||f||_\infty \leq 1, H_\alpha(f) \leq 1\}$$

where

$$H_\alpha(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha},$$

with $0 < \alpha \leq 1$.

Solution: By the Arzela-Ascoli Theorem we need to show that A is closed, bounded, and equicontinous. It is easy to see that the set is bounded. To prove that it is closed, we take a convergent sequence $\{f_n\}$ with limit f and show that $f \in A$. First, note that the uniform limit of continuous functions is continuous and $\|f\|_\infty \leq \|f - f_n\|_\infty + \|f_n\|_\infty \leq \epsilon + 1$ where ϵ can be made arbitrarily small by taking the limit as $n \to \infty$. Thus, $\|f\|_\infty \leq 1$. Moreover, for $x \neq y$

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \leq \frac{\epsilon}{2} + d(x,y)^\alpha + \frac{\epsilon}{2}$$

for n sufficiently large. Which gives that

$$\frac{|f(x) - f(y)|}{d(x,y)^\alpha} \leq \frac{\epsilon}{d(x,y)^\alpha} + 1$$

for arbitrary $\epsilon > 0$. Thus, we conclude that $H_\alpha(f) \leq 1$ and so $f \in A$.

The only thing we have left to show is equicontinuity. Let $\epsilon > 0$. If $d(x,y) \leq \epsilon^{1/\alpha}$ then $d(x,y)^\alpha \leq \epsilon$. Any $f \in A$ satisfies:

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha} \leq 1 \Rightarrow |f(x) - f(y)| \leq d^\alpha(x,y) \leq \epsilon.$$

Thus, we can take $\delta = \epsilon^{1/\alpha}$ to be the equicontinuity contact, which is independent of $f \in A$.

4. (20 points) Define the function $g : (0,1) \to (0,1)$ by

$$g(x) = \begin{cases} 0, & 0 < x < \frac{1}{4}, \\ 2\left(x - \frac{1}{4}\right), & \frac{1}{4} \leq x \leq \frac{3}{4}, \\ 1, & \frac{3}{4} < x < 1. \end{cases}$$

Consider the multiplication operator $M : L^2[(0,1)] \to L^2[(0,1)]$ defined by $M[f](x) = g(x)f(x)$.

(a) (4 points) Find the norm of M.

(b) (8 points) Find the point spectrum of M and describe the eigenspace of each of the eigenvalues in the point spectrum.

(c) (8 points) Find the continuous and residual spectrum of M.

Solution:

(a) We can compute directly

$$\begin{aligned} ||M|| &= \sup_{||f||=1} ||Mf|| \\ &= \sup_{||f||=1} \sqrt{\int_0^1 g^2(x) f^2(x)\, dx} \\ &\leq \sup_{||f||=1} \sqrt{\int_0^1 |f(x)|^2\, dx} \\ &= 1. \end{aligned}$$

Taking any function f with support $[3/4, 1]$ with $||f|| = 1$ we see that $||M[f]|| = 1$ so $||M|| = 1$.

(b) Any non-zero function, f, with support on $[0, 1/3]$ will be such that $Mf = 0 \cdot f$ and thus zero is an eigenvalue with eigenspace:

$$\left\{ f \in L^2([0,1]) :\ f \text{ is non-zero and has support on a subset of } [0, 1/3] \right\}.$$

Similarly, any function with support on $[3/4, 1]$ satisfies $Mf = f$. Thus, one is an eigenvalue with eigenspace:

$$\left\{ f \in L^2([0,1]) :\ f \text{ is non-zero and has support on a subset of } [3/4, 1] \right\}.$$

Thus, $\{0, 1\} \subset \sigma_p(M)$. For general $\lambda \in (0,1)$ we see that if $Mf = gf = \lambda f$ we need $g(x) - \lambda$ to be constant almost everywhere, which we know is not the case! Thus, in fact $\sigma_p(M) = \{0, 1\}$.

(c) Note that $\sigma(M) \subset [-\|M\|, \|M\|] = [-1, 1]$. In fact, $\sigma(M) = \overline{\{g(x) : x \in \mathbb{R}\}} = [0, 1]$. To prove this let $\lambda \notin \overline{\{g(x) : x \in \mathbb{R}\}}$. First, note that λ cannot be in the point spectrum as $g(x) - \lambda$ is bounded away from zero. Thus, for $f \in L^2(\mathbb{R})$ define

$$h(x) = \frac{f(x)}{\lambda - g(x)}$$

so $f(x) = (\lambda - g(x))h(x)$ and so the mapping $G - \lambda I$ is onto. Thus, λ cannot be in the continuous or residual spectrum. Now, if $\lambda \in \overline{\{g(x) : x \in \mathbb{R}\}}$ we claim that $G - \lambda I$ is not onto. Assume that it is, for contradiction. Then $(G - \lambda I)^{-1}$ is bounded by the open mapping theorem. Also, if $G - \lambda I$ is onto then its range is the whole space, a Hilbert Space. Hence, it is closed. Thus, we can invoke Proposition 5.30 once more and note that then:

$$\|(G - \lambda I)h\| \geq c\|h\|, \tag{1}$$

for all $h \in H$. Our goal is to find a function h such that the above inequality breaks down. To do this we take advantage of the singularity at some $x_0 \in \mathbb{R}$, this is because it must be that $\lambda = g(x_0)$ for some $x_0 \in \mathbb{R}$. In fact, then we have that $|\lambda - g(x)| \leq \epsilon$ on some ball of radius r, $B_r(x_0)$. Now, define $f(x) = \frac{1}{\sqrt{2r}}\chi_{B_r(x_0)}$ and let $h = \frac{f(x)}{\lambda - g(x)}$. Note that $\|f\| = 1$ and so $\|h\| \geq \frac{1}{\epsilon}$ and so we have that

$$\|(G - \lambda I)h\| = \|f\| < \frac{c}{\epsilon}$$

for ϵ sufficiently small. There are two possibilities to consider: either $h \notin L^2(\mathbb{R})$ or we have broken inequality (1). Also, $\sigma_r = \emptyset$ as the operator is self-adjoint and thus, $\sigma_c = (0, 1)$.

5. Let f be a non-decreasing function defined on $[0, 1]$.

   (a) (10 points) Prove that
$$\int_0^1 f'(x)\,dx \le f(1) - f(0).$$

   (b) (10 points) Let $\{f_n\}$ be a sequence of non-decreasing functions on $[0, 1]$ such that the series $F(x) = \sum_{n=1}^{\infty} f_n(x)$ converges for all $x \in [0, 1]$. Prove that $F'(x) = \sum_{n=1}^{\infty} f_n'(x)$ almost everywhere.

Solution: (a) We begin by extending the function f such that $f(x) = f(1)$ for all $x > 1$. As f is non-decreasing it is differentiable almost everywhere and so we can write

$$f'(x) = \lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h}$$

for almost all x. Note that the quotient above is non-negative *a.e.* x since f is non-decreasing. We can apply Fatou's Lemma to obtain

$$\begin{aligned}
\int_0^1 f'(x)\,dx &= \int_0^1 \lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h}\,dx \\
&\le \liminf_{h\to 0^+} \frac{1}{h} \int_0^1 f(x+h) - f(x)\,dx \\
&= \liminf_{h\to 0^+} \frac{1}{h} \left(\int_1^{1+h} f(x)\,dx - \int_0^h f(x) \right) \\
&= f(1) - f(0).
\end{aligned}$$

To prove part (b) note that since all $f_n's$ are non-decreasing, then so is F and so it is differentiable almost everywhere. Define

$$R_N(x) = \sum_{n=N+1}^{\infty} f_n'(x)$$

so that $F(x) = \sum_{n=1}^{N} f_n(x) + R_N(x)$ and $F'(x) = \sum_{n=1}^{N} f_n'(x) + R_N'(x)$ for almost all x. we aim to show that $\lim_{N\to\infty} R_N'(x) = 0$. To achieve this note that

$$R_N'(x) - R_{N+1}'(x) = (R_N - R_{N+1})'(x) = f_N'(x) \ge 0$$

as f_N is non-decreasing. From this, we conclude that $\{R_N'\}_N$ is monotone decreasing for almost all x. Thus, $\lim_{N\to\infty} R_N'(x)$ exists for almost all x and is non-negative. To show that it is zero we apply the Monotone Convergence Theorem

$$\int_0^1 \lim_{N\to\infty} R_N'(x)\,dx = \lim_{N\to\infty} \int_0^1 R_N'(x)\,dx \le \lim_{N\to\infty} [R_N(1) - R_N(0)] = 0.$$