**Applied Analysis Preliminary Exam**
9:00-12:00 January 6, 2022

Instructions: You have three hours to complete this exam. Work all five problems; each is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are being asked to prove such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name.

**Problem 1:**

(a) Show that

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{ke^{k/n}}{n^2}$$

exists. For extra credit, what is A?

(b) Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f_1(x) = e^{x_1}\cos(x_2)$, $f_2(x) = e^{x_1}\sin(x_2)$ where $x = (x_1, x_2)$. Use the Inverse Function theorem to show that $f$ is locally invertible.

(c) Is $f$ given in (b) globally invertible? Explain.

(d) Suppose that $g : \mathbb{R} \to \mathbb{R}$ is differentiable and $g'(x) \neq 0$ for all $x \in \mathbb{R}$. Show that $g$ is globally invertible on its range.

*Solution/Hint*:[problem 1]

(a) Note that $ke^{k/n} \leq ke$ whenever $k \leq n$, and so the sum of positive terms is $\leq e(n)(n+1)/2$, and thus $A \leq e/2$ and since the partial sums are monotone increasing and bounded they converge. To compute the limit we write

$$A_n = \frac{d}{d\alpha}\bigg|_{\alpha=1} \frac{1}{n}\sum_{k=1}^{n} e^{\alpha k/n} = \frac{d}{d\alpha}\bigg|_{\alpha=1} \frac{1}{n}\frac{e^{\alpha/n}(1-e^{\alpha})}{1-e^{\alpha/n}}$$

Differentiating and setting $\alpha = 1$ then eventually gives $\lim_{n\to\infty} A_n = 1$.

(b) Note that $f$ is $C^1$ and at an arbitrary point in $\mathbb{R}^2$, the Jacobian

$$Df = \begin{pmatrix} e^{x_1}\cos x_2 & -e^{x_1}\sin x_2 \\ e^{x_1}\sin x_2 & e^{x_1}\cos x_2 \end{pmatrix}$$

so that $\det Df = e^{x_1} > 0$. Thus since $f$ is smooth the Inverse Function Theorem implies there is an open neighborhood $U$ of $(x_1, x_2)$ and $V$ of $f(x_1, x_2)$ so that there is a unique inverse $f^{-1} : V \to U$.

(c) No, since $f$ is not injective, e.g. $f(1, 0) = f(1, 2\pi)$.

(d) Since $g' \neq 0$, then $g$ is monotone, and thus is injective by the intermediate value theorem. Moreover, $g$ is surjective on $g(\mathbb{R})$, so $g : \mathbb{R} \to g(\mathbb{R})$ is bijective.

**Problem 2:**

(a) Consider the nonlinear integral equation

$$f(x) - \frac{1}{10}\int_0^1 (x + y^2)f^2(y)\, dy = \frac{1}{3}.$$

Show that there is a unique continuous solution $f : [0, 1] \to \mathbb{R}$ of this equation with the property that $0 \leq f(x) \leq 1$ for all $x \in [0, 1]$.

(b) Consider the function $f : \mathbb{R}^6 \to \mathbb{R}^2$, with variables $(u, v, w, x, y, z)$ defined by

$$f_1 = u^2 + v^2 + w^2,$$
$$f_2 = xu^2 - yv^2 + zw^2.$$

(1) Find $Df$.

(2) Consider the point $(1,1,1,2,1,-1) \in \mathbb{R}^6$. Let $f(1,1,1,2,1,-1) = f_0$. Show that there exist two functions

$$u : \mathbb{R}^4 \to \mathbb{R}, \quad v : \mathbb{R}^4 \to \mathbb{R},$$

of the four variables $(w, x, y, z)$ that are continuously differentiable on some ball $B$ centered at the point $(w, x, y, z) = (1, 2, 1, -1)$, such that $u(1, 2, 1, -1) = 1$, $v(1, 2, 1, -1) = 1$, and the equations $f(u, v, w, x, y, z) - f_0 = 0$ both hold for all $(w, x, y, z) \in B$.

(3) Can the implicit function theorem be applied at the same point to find functions $(v, w)$ of the variables $(u, x, y, z)$ that satisfy $f - f_0 = 0$? Why or why not?

*Solution/Hint*:

(a) We rewrite this problem as a fixed-point problem. Consider

$$Tf = \frac{1}{3} + \frac{1}{10}\int_0^1 (x + y^2) f^2(y)\, dy.$$

- We first note that $T : C([0,1]) \to C([0,1])$. Indeed, since $k(x, y) = (x + y^2)$ is a continuous function on $[0,1] \times [0,1]$ we obtain that for $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|k(x_1, y_1) - k(x_2, y_2)| < \epsilon \quad \text{if} \quad \|(x_1, y_1) - (x_2, y_2)\|_1 < \delta.$$

Let $g(x) = \int_0^1 k(x, y)\, dy$ then $g$ is continuous:

$$|g(x_1) - g(x_2)| = \int_0^1 |k(x_1, y) - k(x_2, y)|\, dy < \epsilon$$

provided $|x_1 - x_2| < \delta$ Thus, if $f \in C([0,1])$ then we have that $Tf \in C([0,1])$.

- The problem hints that we should look at the set

$$X = \{f \in C([0,1] : \ 0 \le f(x) \le 1 \quad \text{for all} \quad x \in [0,1]\}.$$

Let us now prove that $T : X \to X$. We can see that $Tf \ge \frac{1}{3}$, moreover, we can see that if $0 \le f(x) \le 1$ for all $x \in [0,1]$ then

$$T(f) \le \frac{1}{3} + \frac{1}{10}\int_0^1 (x + y^2)\, dy = \frac{1}{3} + \frac{1}{10}\left[xy + \frac{1}{3}y^3\right]_0^1 \le \frac{7}{15}.$$

Thus, $T : X \to X$.

- We now show that $T$ is a contraction. Indeed, we have that

$$\begin{aligned}
|T(f) - T(g)| &\le \frac{1}{10}\int_0^1 (x + y^2)(f^2(y) - g^2(y))\, dy \\
&\le 2\frac{1}{10}\int_0^1 (x + y^2)\, dy \|f - g\|_\infty \\
&= \frac{1}{15}\|f - g\|_\infty.
\end{aligned}$$

Hence, $T$ is a contraction.

- $X$ is a closed subset of a complete metric space, thus it is a complete metric space and by the CMT we see that there exists a unique $f$ that solve our integral equation.

(b) Let $p = (1,1,1,2,1,-1)$. Note that $f(p) = (3,0) = f_0$.

(1) $Df = \begin{pmatrix} 2u & 2v & 2w & 0 & 0 & 0 \\ 2xu & -2yv & 2zw & u^2 & -v^2 & w^2 \end{pmatrix}$

(2) If we think of $(u, v)$ as the variables then

$$D_{u,v} f|_p = \begin{pmatrix} 2 & 2 \\ 4 & -2 \end{pmatrix}$$

which is nonsingular. Thus the implicit function theorem applies to give $(u, v) : \mathbb{R}^4 \to \mathbb{R}^2$ on a ball $B_{(1,2,1,-1)}$ in $\mathbb{R}^4$ satisfying $f(u(w, x, y, z), v(w, x, y, z); w, x, y, z) = (3, 0)$ with the restriction that at $(1, 2, 1, -1)$, $u = 1$ and $v = 1$

(3) The implicit function theorem does not apply to find $v(u,x,y,z)$ and $w(u,x,y,z)$ because the Jacobian

$$D_{v,w}f|_p = \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}$$

is singular.

**Problem 3:** Let $X$ and $Y$ be Hilbert spaces and $T : X \to Y$ be a bounded linear operator. Prove that if $T$ is compact then its adjoint $T^* : Y^* \to X^*$ is also compact.

*Solution/Hint*: Let $\{y_n^*\}_{n\geq 1}$ be a sequence in $Y^*$ with $\|y_n^*\| \leq 1$. We want to show that the sequence $\{T^* y_n^*\}_{n\geq 1}$ has a convergent subsequence. Let $B_1$ be the closed unit ball in $X$, since $T$ is compact we know that $E := \overline{T(B_1)} \subset Y$ is compact.

Now, by definition $y_n^*$ is a bounded linear functional from $Y \to \mathbb{R}$. Let $f_n : E \to \mathbb{R}$ be the restriction of $y_n^*$ to $E$. We will show that $\{f_n\}_{n\geq 1}$ satisfies the hypothesis of the Arezela-Ascoli Theorem. We first show that they are uniformly Lipschitz continuous. Indeed, note that:

$$\left|f_n(y) - f_n(y')\right| \leq \|y_n^*\|\|y - y'\| \leq \|y - y'\|,$$

for all $y, y' \in E$. Thus, the sequence is equicontinuous. Also, note that

$$\sup_{y\in E} \|y\| = \sup_{\|x\|\leq 1} \|Tx\| = \|T\|$$

thus;

$$|f_n(y)| \leq \|y_n^*\|\|y\| \leq \|T\|.$$

Hence, the sequence $|f_n(y)|$ are uniformly bounded. Applying Arezela-Ascoli we see that there exists a subsequence $\{f_{n_j}\}_{j\geq 1}$, which converges uniformly to a function $f \in E$.

To finish the proof observe that

$$\begin{aligned} \|T^* y_{n_i}^* - T^* y_{n_j}^*\| &= \sup_{\|x\|\leq 1} \left|\left(T^* y_{n_i}^* - T^* y_{n_j}^*, x\right)\right| \\ &= \sup_{\|x\|\leq 1} \left|\left(y_{n_i}^* - y_{n_j}^*, Tx\right)\right| \\ &= \sup_{\|x\|\leq 1} \|f_{n_i}(Tx) - f_{n_j}(Tx)\|. \end{aligned}$$

However, the right hand side of the above inequality approaches zero as $i, j \to \infty$. Thus, the sequence is Cauchy and converges to an element $x^* \in X^*$. Thus, we can conclude that $T^*$ is compact.

**Problem 4:** Let $D$ be a countable set. Prove that any sequence of functions $f_n : D \to E$ such that the set $\{f_n(d)\}_{n=1}^{\infty}$ is precompact for each $d \in D$ has a subsequence which is point-wise convergent in $D$.

*Solution/Hint*: Let $D = \{d_1, d_2, \cdots\}$. Note that the set $\{f_n(d_1)\}_{n\geq 1}$ is precompact in $E$. Thus there is a convergent subsequence $\{f_n(d_1)\}_{n\in N_1}$, where $N_1 \subset \mathbb{N}$, call its limit $f(d_1)$. Again $\{f_n(d_2)\}_{n\in N_1}$ is precompact in $E$ and there is a convergent subsequence $\{f_n(d_2)\}_{n\in N_2}$, where $N_2 \subset N_1$, call its limit $f(d_2)$.

Proceeding in this fashion we know that $\{f_n(d_k)\}_{n\in N_{k-1}}$ is precompact in $E$ and there exists a convergent subsequence $\{f_n(d_k)\}_{n\in N_k}$, where $N_k \subset N_{k-1}$, call its limit $f(d_k)$. Let the set $N \subset N_1$ whose $j^{th}$ element is the $j^{th}$ element of $N_j$. The sequence $\{f_n(d_i)\}_{n\in N}$ converges to $f(d_i)$ for all $i \in \mathbb{N}$. For full credit, prove the last claim rigorously.

**Problem 5:** Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function. Prove that $f$ maps sets of Lebesgue measure zero to sets of Lebesgue measure zero.

*Solution/Hint*: Let $A \subset [a,b]$ be an arbitrary set of Lebesgue measure zero. Our goal is to show that for arbitrary $\epsilon > 0$ we have $\mu(f(A)) < \epsilon$, where $\mu$ represents the Lebesgue measure. To prove this we use the absolute continuity of $f$. That is, choose $\delta > 0$ such that for any finite family of intervals $[a_i, b_i]$ with total length is less than $\delta$, *i.e* $\sum_{i=1}^{N}(b_i - a_i) < \delta$, we have that

$$\sum_{i=1}^{N} |f(b_i) - f(a_i)| < \epsilon.$$

Now, given that $\mu(A) = 0$ there exists an open set $A \subset U$ such that $\mu(U) < \delta$. Since $U$ is an open subset of $\mathbb{R}$, we can express it as a countable union of disjoint open intervals $U = \cup_{i \geq 1}(c_i, d_i)$. Now we can obtain an upper bound:

$$\mu(f(A)) \leq \mu(f(V)) \leq \sup_n \sum_{k=1}^{n} \mu(f([c_k, d_k])) = \sup_n \sum_{k=1}^{n} |f(M_k) - f(m_k)|,$$

where $f$ attains its maximum and minimum in the interval $[c_k, d_k]$ on $M_k$ and $m_k$ respectively. However, we have that

$$\sum_{k=1}^{n} |M_k - m_k| \leq \sum_{k=1}^{n} (d_k - c_k) < \delta$$

and thus we can conclude that

$$\mu(f(A)) < \epsilon.$$