Applied Analysis Preliminary Exam (Hints/solutions)
9:00 AM – 12:00 PM, Monday August 19, 2024

Instructions You have three hours to complete this exam. Work all five problems; there are no optional problems. Each problem is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are directly proving such a theorem (when in doubt, ask the proctor). If you cannot finish part of a question, you may wish to move on to the next part; problems are graded with partial credit. **Write your student number on your exam, not your name**.

Problem 1 (20 points) Let $\{f_n\}$ be a sequence of nondecreasing functions that map $[0,1]$ to itself, i.e., $\forall n \in \mathbb{N}$, $x \geq y \implies f_n(x) \geq f_n(y)$. Let $g : [0,1] \longrightarrow [0,1]$ be a continuous function on $[0,1]$ such that

$$\forall x \in [0,1], \quad \lim_{n\to\infty} f_n(x) = g(x). \tag{1}$$

Note that the f_n need not be continuous.

(a) Let $0 \leq a < b \leq 1$ and (just for this subproblem) suppose there is some $\varepsilon > 0$ such that $\forall x, y \in [a,b]$ then $|g(x) - g(y)| < \varepsilon$. Prove that for n sufficiently large, then $\forall x, y \in [a,b]$ then $|f_n(x) - f_n(y)| < 3\varepsilon$. *Hint: assuming $x \leq y$, prove that for n sufficiently large,*

$$\forall x \in [a,b], \qquad g(a) - \varepsilon < f_n(a) \leq f_n(x) \leq f_n(y) \leq f_n(b) < g(b) + \varepsilon < g(a) + 2\varepsilon. \tag{2}$$

(b) Prove that f_n converges uniformly on $[0,1]$ to g.

(c) Give an example to show that the convergence need not be uniform if we no longer assumed g is continuous.

Solution: By Heine's theorem, g is uniformly continuous on $[0,1]$. Let $\varepsilon > 0$, there exists $\delta > 0$, such that

$$|x - y| < \delta \Rightarrow |g(x) - g(y)| < \varepsilon/3. \tag{3}$$

Now, we lay down a uniform grid on $[0,1]$, defined by

$$[0,1] = \bigcup_{k=1}^{N} [\xi_{k-1}, \xi_k], \qquad \text{with} \quad \xi_k = k/N, \text{and } 1/N < \delta. \tag{4}$$

We evaluate the f_n and g on the grid $\{\xi_k\}$; since f_n converges to g pointwise at every point of the finite grid, there exists M, such that

$$\forall n \geq M, \ \forall 1 \leq k \leq N, \qquad \Big| g(\xi_k) - f_n(\xi_k) \Big| < \varepsilon/3. \tag{5}$$

Finally we tie all the points in a grid cell to the value of f_n at that grid point. We have $\forall x \in [\xi_{k-1}, \xi_k]$,

$$g(\xi_{k-1}) - \varepsilon/3 < f_n(\xi_{k-1}) < f_n(x) < f_n(\xi_k) < g(\xi_k) + \varepsilon/3 < g(\xi_{k-1}) + 2\varepsilon/3, \tag{6}$$

where the first and fourth inequalities are a consequence of (5); the second and third inequality are consequences of the monotonicity of f_n; and the last inequality is a consequence of (3). Therefore,

$$\left|g(\xi_{k-1}) - f_n(x)\right| < 2\varepsilon/3. \tag{7}$$

Whence we conclude that

$$\left|g(x) - f_n(x)\right| \leq \left|g(x) - g(\xi_{k-1})\right| + \left|g(\xi_{k-1}) - f_n(x)\right| \leq \varepsilon \tag{8}$$

Since the left hand side does not depend on $x \in [0,1]$, we can upgrade the pointwise convergence of f_n to uniform convergence.

Problem 2 (20 points) Let $\mathscr{H}$ be a separable complex Hilbert space, and let $\mathcal{B}(\mathscr{H})$ be the Banach space of bounded linear operators from $\mathscr{H}$ to itself equipped with the operator norm. As is customary, we use the same notation to denote the operator norm (as in $\|T\| = \|T\|_{\mathcal{B}(\mathscr{H})}$) and the norm induced by the inner product in $\mathscr{H}$ (as in $\|x\|^2 = \langle x, x\rangle$).

We denote by $\mathscr{K}(\mathscr{H}) \subset \mathcal{B}(\mathscr{H})$ the closed subspace of *compact* operators, and by $\mathscr{F}(\mathscr{H})$ the subspace of *finite rank* operators. We recall that an operator $T \in \mathcal{B}(\mathscr{H})$ is finite rank if its range has finite rank. All finite rank operators are compact; the limit of a convergent sequence of finite rank operators is therefore compact.

(a) Prove that $\mathscr{F}(\mathscr{H})$ is dense everywhere in $\mathscr{K}(\mathscr{H})$.

(b) Let $\left\{f_n\right\}$ be an orthonormal family in $\mathscr{H}$. Prove that

$$T \in \mathscr{K}(\mathscr{H}) \Longrightarrow \lim_{n\to\infty} Tf_n = 0. \tag{9}$$

You may use the following result (which we proved in class): a sequence $\left\{z_n\right\}$ defined in a compact metric space converges to a limit z if and only if all convergent subsequences converge to z.

(c) Let $\{e_n\}$ be an orthonormal basis for $\mathscr{H}$. Prove that

$$\sum_{n=0}^{\infty} \|Te_n\|^2 < \infty \Longrightarrow T \in \mathscr{K}(\mathscr{H}). \tag{10}$$

Solution:

(a) Let $T \in \mathscr{K}(\mathscr{H})$ and let $\varepsilon > 0$, we will exhibit a finite rank operator T_ε such that $\left\|T - T_\varepsilon\right\| < \varepsilon$. Let $B(0,1)$ be the unit ball in $\mathscr{H}$. T being compact, $T(B(0,1))$ has compact closure, so it is totally bounded, and $\exists N(\varepsilon)$, $\exists y_1, \ldots, y_{N(\varepsilon)}$, such that

$$T(B(0,1)) \subset \bigcup_{1}^{N(\varepsilon)} B(y_j, \varepsilon). \tag{11}$$

Now, define $V = \operatorname{span}\left\{y_1, \ldots, y_{N(\varepsilon)}\right\}$, and let P_V be the orthogonal projector onto V. The operator

$$T_\varepsilon \stackrel{\text{def}}{=} P_V \circ T, \tag{12}$$

has finite rank ($\leq N(\varepsilon)$). Let $x \in B(0,1)$, because of (11), $\exists y_j$ such that $\|Tx - y_j\| \leq \varepsilon$. But $y_j \in V$, and therefore $\operatorname{dist} Tx, V \leq \varepsilon$. We conclude that $\|Tx - P_V Tx\| \leq \varepsilon$, and thus

$$\forall x \in B(0,1), \qquad \left\|(T - T_\varepsilon)(x)\right\| \leq \varepsilon. \tag{13}$$

Whence $\|T - T_\varepsilon\| \leq \varepsilon$.

(b) Because T is compact, $T(B(0,1))$ has compact closure and therefore the sequence $\left\{Tf_n\right\}$ admits a convergent subsequence. Without loss of generality, we denote by $\left\{Tf_n\right\}$ this subsequence, which converges to $y \in \overline{T(B(0,1))}$.

Let $x \in \mathscr{H}$, we have $\langle Tf_n, x\rangle = \langle f_n, T^*x\rangle$. Also, since $\left\{f_n\right\}$ is an orthonormal family, it converges weakly to 0,

$$\lim_{n\to\infty} \langle f_n, T^*x\rangle = 0, \tag{14}$$

and therefore

$$\lim_{n\to\infty} \langle Tf_n, x\rangle = 0 = \langle y, x\rangle. \tag{15}$$

We conclude that $y = 0$. To wit, any convergent subsequence of $\left\{Tf_n\right\}$ converges to the same limit 0. We conclude that $\lim_{n\to\infty} Tf_n = 0$.

(c) Let $T \in \mathcal{B}(\mathscr{H})$ such that $\sum_{n=0}^{\infty} \|Te_n\|^2 < \infty$. We will prove that T is the limit of a sequence of finite rank operators. We consider the sequence of truncated operators defined by their action on the basis $\left\{e_n\right\}$,

$$T_N x = \sum_{n=0}^{N} x_n T e_n = \sum_{n=0}^{N} \langle x, e_n\rangle T e_n. \tag{16}$$

T_N is a finite rank operator in $\mathcal{B}(\mathscr{H})$. We verify that T_N converges to T in $\mathcal{B}(\mathscr{H})$. We have

$$\left\|\left(T - T_N\right)x\right\| = \Big\| \sum_{n=N+1}^{\infty} \langle x, e_n\rangle T e_n \Big\| \leq \sum_{n=N+1}^{\infty} \left|\langle x, e_n\rangle\right| \left\|Te_n\right\| \tag{17}$$

$$\leq \left[\sum_{n=N+1}^{\infty} \left|\langle x, e_n\rangle\right|^2\right]^{1/2} \left[\sum_{n=N+1}^{\infty} \left\|Te_n\right\|^2\right]^{1/2} \leq \|x\| \left[\sum_{n=N+1}^{\infty} \left\|Te_n\right\|^2\right]^{1/2}, \tag{18}$$

whence

$$\left\|\left(T - T_N\right)\right\| \leq \left[\sum_{n=N+1}^{\infty} \left\|Te_n\right\|^2\right]^{1/2}. \tag{19}$$

We conclude by observing that $\lim_{N\to\infty} \sum_{n=N+1}^{\infty} \left\|Te_n\right\|^2 = 0$, since it is the tail of a convergent series, and thus T_N converges to T in $\mathcal{B}(\mathscr{H})$. T is therefore compact, since it is the limit of finite rank operators.

See also Theorem 9.21 in Hunter and Nachtergaele

Problem 3 (20 points)

(a) Let $\mathbb{T}$ be the standard 1D torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. For which natural numbers k is $C^k(\mathbb{T})$ dense in $L^2(\mathbb{T})$ (that is, with respect to the norm on $L^2(\mathbb{T})$)? Justify your answer.

Solution: It's dense for $\boxed{\text{all } k \in \mathbb{N}}$.

You could do this by mollifying a function, or you could use Fourier series. With Fourier series, if we truncate them to be finite, then we're a trigonometric polynomial (a finite sum of C^∞ functions) and hence C^∞, and since the Fourier series tail converges, we have the density.

(b) Suppose $f \in L^2(\mathbb{T})$ has the property that $\exists M$ such that

$$\forall \varphi \in C^1(\mathbb{T}) \quad \left| \int_{\mathbb{T}} f\varphi' \right| \le M\|\varphi\|_{L^2}.$$

Prove that there exists a function g such that (1) $g \in L^2(\mathbb{T})$, and (2)

$$\forall \varphi \in C^1(\mathbb{T}) \quad \int_{\mathbb{T}} g\varphi = -\int_{\mathbb{T}} f\varphi'.$$

Solution: By the assumption, we have that the linear functional $\varphi \mapsto -\int_T f\varphi'$ is bounded (it's obviously linear since both differentiation and integration are linear operators). Thus via the $\boxed{\text{Bounded Linear Transformation}}$ theorem, since $C^1(\mathbb{T})$ is dense in $L^2(\mathbb{T})$, we can extend this functional to be defined on all of $L^2(\mathbb{T})$. Then since this is a bounded linear functional on a Hilbert space, we can identify it with an element g inside the same Hilbert space using the $\boxed{\text{Riesz Representation}}$ theorem.

This is discussed in the Hunter and Nachtergaele book in between Definition 7.6 and 7.7, though it's not explicitly proved (but it is outlined).

Problem 4 (20 points) Let M be a linear subspace of a normed linear space X, and let $x_0 \in X$ with $\operatorname{dist}(x_0, M) = \delta > 0$ where $\operatorname{dist}(x, V) \stackrel{\text{def}}{=} \inf_{v\in V} \|x - v\|$. Prove that there is a bounded linear functional $\varphi \in X^*$ such that $\|\varphi\| \le \delta^{-1}$, $\varphi(x_0) = 1$, and $\varphi(m) = 0\ \forall m \in M$.

Solution: The key trick to this problem is appropriately using the $\boxed{\text{Hahn-Banach theorem}}$. There are several standard variants of the Hahn-Banach theorem. Here's the one from Hunter and Nachtergaele (Thm. 5.58) that will be sufficient for our purposes: *if Y is a linear subspace of a normed linear space X and $\psi : Y \to \mathbb{R}$ is a bounded linear functional on Y, then ψ can be extended to a bounded linear functional φ on all of X that has the same norm.*

A common corollary (exercise 5.6 in Hunter and Nachtergaele, or theorem 4.3-3 in Kreyszig) is that for any $0 \ne x_0 \in X$ there is a $\varphi \in X^*$ that has unit norm and $\varphi(x_0) = \|x_0\|$. This problem is a generalization of that.

Let's start by considering $\delta = 0$. In that case, we can simply define $\varphi = 0$ and clearly this satisfies the requirements.

Now assume $\delta > 0$. Let $N = \operatorname{span}(x_0) = \{\alpha x_0 \mid \alpha \in \mathbb{R}\}$ which is a subspace. Then let $Y = M + N$ which is obviously a subspace, and in fact we can write it as the direct sum

$Y = M \oplus N$ meaning that any decomposition $y = m + n$ with $m \in M, n \in N$ is unique. This is true if $M \cap N = \{0\}$ (cf. section 8.1 in Hunter and Nachtergaele; quick proof via contradiction), which is true in our case since $\delta > 0$.

Recalling that if $n \in N$ then we can (uniquely) write $n = \alpha x_0$, we define ϕ on Y as

$$\phi(y = m + \alpha x_0) = \alpha$$

and note that $\phi(x_0) = 1$. Consider $y' = m' + \alpha' x_0$, then $y + y' = (m + m') + (\alpha + \alpha')x_0$ and since $m + m' \in M$ since it's a subspace, we have $\phi(y + \beta y') = \alpha + \beta\alpha' = \phi(y) + \beta\phi(y')$ showing that ϕ is linear. Is it bounded? Yes. To show this, we want to show $\exists C \stackrel{\text{def}}{=} \|\phi\|$ such that for all $y \in Y$ we have $|\phi(y)| \leq C\|y\|$. So pick any nonzero $y = m + \alpha x_0 \in Y$. Then either $\alpha = 0$ which gives the result, or else we can write

$$\begin{aligned}
\frac{|\phi(y)|}{\|y\|} &= \frac{|\alpha|}{\|y\|} \\
&= \frac{|\alpha|}{|\alpha| \cdot \|\alpha^{-1} y\|} \\
&= \frac{1}{\|\alpha^{-1} m + x_0\|} \\
&\leq \delta^{-1} = C
\end{aligned}$$

since $\|\alpha^{-1} m + x_0\| \geq \operatorname{dist}(x_0, M) = \delta$. Hence $\|\phi\| = \delta^{-1}$ showing that ϕ is bounded on Y. To finish, we invoke the Hahn-Banach theorem to extend ϕ to some φ on all of X.

Problem 5 (20 points)

Let $\Omega = [0, 1]$ and let the kernel $\kappa : \Omega \times \Omega \to \mathbb{C}$ be a symmetric function, meaning that $\kappa(x, y) = \overline{\kappa(y, x)}$.

Assume that κ is *positive-definite* meaning that for all $n \in \mathbb{N}$ and all complex numbers $(c_1, \ldots, c_n)$ and points $(x_1, \ldots, x_n) \subset \Omega$, then

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{c_i} c_j \kappa(x_i, x_j) \geq 0.$$

In other words, the matrix $K^{(n)}$ with entries $K^{(n)}_{i,j} = \kappa(x_i, x_j)$ is a Hermitian positive semi-definite matrix. *Note: you may assume all fields are real rather than complex if you wish.*

We assume κ is (jointly) continuous and not identically equal to 0. Let $\mathscr{H} \stackrel{\text{def}}{=} L^2(\Omega)$. Associated with κ is the linear operator $\mathcal{T}_\kappa : \mathscr{H} \to \mathscr{H}$ defined by $\Big(\mathcal{T}_\kappa f\Big)(x) = \int_\Omega \kappa(x, y) f(y)\, dy$. We'll prove parts of the classic *Mercer's Theorem*.

(a) Prove that $\mathcal{T}_\kappa$ has at least one nonzero eigenvalue.

Solution: First, observe that $\mathcal{T}_\kappa$ defines a linear, self-adjoint operator (self-adjointness follows from the symmetry of κ) on a Hilbert space, and furthermore, since κ is continuous on the compact domain $\Omega \times \Omega$, it follows $\kappa \in L^2(\Omega \times \Omega)$, which means that $\mathcal{T}_\kappa$ is a Hilbert-Schmidt operator (Example 9.23 in Hunter and Nachtergaele), and therefore compact (Thm 9.21). Thus the spectral theorem applies, which guarantees that $\mathscr{H}$ has an orthonormal basis of eigenvectors of $\mathcal{T}_\kappa$. If all these eigenvectors have an eigenvalue of 0, then $\mathcal{T}_\kappa$ is the zero operator, but since κ is not identically zero, $\mathcal{T}_\kappa \neq 0$ so this is a contradiction. Hence at least one eigenvalue is nonzero.

(b) Prove that all eigenvalues of $\mathcal{T}_\kappa$ are nonnegative.

Solution: Let e be an eigenvalue of $\mathcal{T}_\kappa$ with eigenvalue λ, so $\mathcal{T}_\kappa e = \lambda e$. Since κ is continuous, it follows that all functions in the range of $\mathcal{T}_\kappa$ are continuous, and in particular e is continuous.

Taking the inner product of the equation $\mathcal{T}_\kappa e = \lambda e$ with e we see $\lambda\langle e, e\rangle = \langle e, \mathcal{T}_\kappa e\rangle$ and since $\langle e, e\rangle > 0$, it suffices to prove that $\langle e, \mathcal{T}_\kappa e\rangle \geq 0$. In order to exploit the positiveness of κ, we're going to do something like Riemann sums. For any $n \in \mathbb{N}$ define

$$e_n(x) = e\left(\frac{\lfloor nx \rfloor}{n}\right), \quad \text{i.e.,} \quad e_n = \sum_{j=1}^{n} e(x_j)\chi_{\left[\frac{j-1}{n}, \frac{j}{n}\right)} \text{ where } x_j^{(n)} = j/n$$

where $\lfloor x \rfloor$ is the floor operator (the largest integer less than or equal to x) and χ_A is the standard indicator function. We defined e_n in this way so that $\int_0^1 e_n$ is the left Riemann sum.

Note that $e_n \to e$ pointwise due to the continuity of e.

Let $f = \mathcal{T}_\kappa e$ (i.e., $f = \lambda e$) so f is also continuous and we can define $f_n = \lambda e_n$ and $f_n \to f$ pointwise as well. We can also do the same for κ, defining $\kappa_n(x, y) = \kappa\left(x, \frac{\lfloor ny \rfloor}{n}\right)$, which also converges pointwise. Now we claim that for any $x \in \Omega$,

$$\begin{aligned}
f(x) &\stackrel{\text{def}}{=} \int \kappa(x, y)e(y)\, dy \\
&= \int \lim_{m\to\infty} \kappa_m(x, y)e_m(y)\, dy \\
&= \lim_{m\to\infty} \int \kappa_m(x, y)e_m(y)\, dy \quad \text{via the Lebesgue DCT} \\
&= \lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} \kappa\left(x, x_j^{(m)}\right) e(x_j^{(m)})
\end{aligned}$$

where the Dominated Convergence Theorem (DCT) applied because $y \mapsto |\kappa_m(x, y)e_m(y)|$ is uniformly (in m and y) bounded by $\|\kappa_m(x, \cdot)e_m(\cdot)\|_\infty$ and the integral of this is finite since these are continuous functions on a compact domain.

Then

$$
\begin{aligned}
\langle e, \mathcal{T}_\kappa e\rangle = \langle e, f\rangle &= \int \overline{e(x)} f(x)\,dx \\
&= \int \lim_{n\to\infty} \overline{e_n(x)} f_n(x)\,dx \\
&= \lim_{n\to\infty} \int \overline{e_n(x)} f_n(x)\,dx \quad \text{via DCT} \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \overline{e(x_i^{(n)})} f(x_i^{(n)}) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \overline{e(x_i^{(n)})} \lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} \kappa\left(x_i^{(n)}, x_j^{(m)}\right) e(x_j^{(m)}) \\
&= \lim_{n\to\infty} \lim_{m\to\infty} \frac{1}{mn} \sum_{i=1}^{n}\sum_{j=1}^{m} \kappa\left(x_i^{(n)}, x_j^{(m)}\right) \overline{e(x_i^{(n)})} e(x_j^{(m)})
\end{aligned}
$$

Finally, we note that $(x,y) \mapsto \kappa(x,y)\overline{e(x)}e(y)$ is jointly continuous on a compact domain $\Omega \times \Omega$ so uniformly bounded, hence integrable, i.e., $\int\int \left|\kappa(x,y)\overline{e(x)}e(y)\right| < \infty$. As this is a continuous function (absolute value is continuous, so it's the composition of two continuous functions), it is Riemann integrable also. Hence any Riemann sum (that has the spacing of nodes go to zero) converges. That means that

$$
\lim_{n\to\infty} \lim_{m\to\infty} \left| \frac{1}{mn} \sum_{i=1}^{n}\sum_{j=1}^{m} \kappa\left(x_i^{(n)}, x_j^{(m)}\right) \overline{e(x_i^{(n)})} e(x_j^{(m)}) \right|
$$

is bounded (uniformly in m and n), hence we can take a discrete version of Fubini's theorem (cf. Example 12.42 in Hunter and Nachtergaee), i.e., using the counting measure, and therefore write

$$
\begin{aligned}
\lim_{n\to\infty} \lim_{m\to\infty} \frac{1}{mn} \sum_{i=1}^{n}\sum_{j=1}^{m} \kappa\left(x_i^{(n)}, x_j^{(m)}\right) \overline{e(x_i^{(n)})} e(x_j^{(m)}) &= \lim_{m,n\to\infty} \frac{1}{mn} \sum_{i=1}^{n}\sum_{j=1}^{m} \kappa\left(x_i^{(n)}, x_j^{(m)}\right) \overline{e(x_i^{(n)})} e(x_j^{(m)}) \\
&= \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n}\sum_{j=1}^{n} \kappa\left(x_i^{(n)}, x_j^{(n)}\right) \overline{e(x_i^{(n)})} e(x_j^{(n)}) \\
&\geq \lim_{n\to\infty} 0 \\
&\geq 0
\end{aligned}
$$

using the positivity.