Instructions: You have three hours to complete this exam. Work all five problems; each is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are being asked to prove such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name.

Student Number:

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1. (20 points) Compute the following quantities.

   (a) For each real x, compute $\lim_{n\to\infty} e^{-nx}\left(1+\frac{x}{n}\right)^{n^2}$ (Hint: $\ln(1+x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$ for $-1 < x \leq 1$)

   (b) Evaluate $\lim_{n\to\infty}\left[\frac{1}{\sqrt{n(n+0)}} + \frac{1}{\sqrt{n(n+1)}} + \cdots \frac{1}{\sqrt{n(n+n)}}\right]$

2. (20 points) Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction, with contraction constant c. Choose $x_0 \in X$ and define the sequence $\{x_n\}$ by $Tx_n = x_{n+1}$. From the Contraction Mapping Theorem, T has a unique fixed point x. Prove the following inequalities:

   (a) For $n \geq m \geq 1$, $d(x_m, x_n) \leq \frac{c^m}{1-c} d(x_1, x_0)$

   (b) $d(x_m, x) \leq \frac{c^m}{1-c} d(x_1, x_0)$

   (c) $d(x_m, x) \leq \frac{c}{1-c} d(x_{m-1}, x_m)$

3. (20 points) Let (X, d) be a compact metric space. Prove that the following set is compact

$$A = \{f \in C(X) : ||f||_\infty \leq 1, H_\alpha(f) \leq 1\}$$

   where

$$H_\alpha(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha},$$

   with $0 < \alpha \leq 1$.

4. (20 points) Define the function $g : (0,1) \to (0,1)$ by

$$g(x) = \begin{cases} 0, & 0 < x < \frac{1}{4}, \\ 2\left(x - \frac{1}{4}\right), & \frac{1}{4} \leq x \leq \frac{3}{4}, \\ 1, & \frac{3}{4} < x < 1. \end{cases}$$

Consider the multiplication operator $M : L^2[(0,1)] \to L^2[(0,1)]$ defined by $M[f](x) = g(x)f(x)$.

(a) (4 points) Find the norm of M.

(b) (8 points) Find the point spectrum of M and describe the eigenspace of each of the eigenvalues in the point spectrum.

(c) (8 points) Find the continuous and residual spectrum of M.

5. Let f be a non-decreasing function defined on $[0,1]$.

   (a) (10 points) Prove that
   $$\int_0^1 f'(x)\,dx \le f(1) - f(0).$$

   (b) (10 points) Let $\{f_n\}$ be a sequence of non-decreasing functions on $[0,1]$ such that the series $F(x) = \sum_{n=1}^{\infty} f_n(x)$ converges for all $x \in [0,1]$. Prove that $F'(x) = \sum_{n=1}^{\infty} f_n'(x)$ almost everywhere.