Applied Analysis Preliminary Exam
9:00 AM – 12:00 PM, Monday August 19, 2024

Instructions You have three hours to complete this exam. Work all five problems; there are no optional problems. Each problem is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are directly proving such a theorem (when in doubt, ask the proctor). If you cannot finish part of a question, you may wish to move on to the next part; problems are graded with partial credit. **Write your student number on your exam, not your name**.

Problem 1 (20 points) Let $\left\{f_n\right\}$ be a sequence of nondecreasing functions that map $[0,1]$ to itself, i.e., $\forall n \in \mathbb{N}$, $x \geq y \implies f_n(x) \geq f_n(y)$. Let $g : [0,1] \longrightarrow [0,1]$ be a continuous function on $[0,1]$ such that

$$\forall x \in [0,1], \quad \lim_{n\to\infty} f_n(x) = g(x). \tag{1}$$

Note that the f_n need not be continuous.

(a) Let $0 \leq a < b \leq 1$ and (just for this subproblem) suppose there is some $\varepsilon > 0$ such that $\forall x, y \in [a,b]$ then $|g(x)-g(y)| < \varepsilon$. Prove that for n sufficiently large, then $\forall x, y \in [a,b]$ then $|f_n(x) - f_n(y)| < 3\varepsilon$. *Hint: assuming $x \leq y$, prove that for n sufficiently large,*

$$\forall x \in [a,b], \qquad g(a) - \varepsilon < f_n(a) \leq f_n(x) \leq f_n(y) \leq f_n(b) < g(b) + \varepsilon < g(a) + 2\varepsilon. \tag{2}$$

(b) Prove that f_n converges uniformly on $[0,1]$ to g.

(c) Give an example to show that the convergence need not be uniform if we no longer assumed g is continuous.

Problem 2 (20 points) Let $\mathscr{H}$ be a separable complex Hilbert space, and let $\mathcal{B}(\mathscr{H})$ be the Banach space of bounded linear operators from $\mathscr{H}$ to itself equipped with the operator norm. As is customary, we use the same notation to denote the operator norm (as in $\|T\| = \|T\|_{\mathcal{B}(\mathscr{H})}$) and the norm induced by the inner product in $\mathscr{H}$ (as in $\|x\|^2 = \langle x, x\rangle$).

We denote by $\mathscr{K}(\mathscr{H}) \subset \mathcal{B}(\mathscr{H})$ the closed subspace of *compact* operators, and by $\mathscr{F}(\mathscr{H})$ the subspace of *finite rank* operators. We recall that an operator $T \in \mathcal{B}(\mathscr{H})$ is finite rank if its range has finite rank. All finite rank operators are compact; the limit of a convergent sequence of finite rank operators is therefore compact.

(a) Prove that $\mathscr{F}(\mathscr{H})$ is dense everywhere in $\mathscr{K}(\mathscr{H})$.

(b) Let $\left\{f_n\right\}$ be an orthonormal family in $\mathscr{H}$. Prove that

$$T \in \mathscr{K}(\mathscr{H}) \Longrightarrow \lim_{n\to\infty} T f_n = 0. \tag{3}$$

You may use the following result (which we proved in class): a sequence $\{z_n\}$ defined in a compact metric space converges to a limit z if and only if all convergent subsequences converge to z.

(c) Let $\{e_n\}$ be an orthonormal basis for $\mathscr{H}$. Prove that

$$\sum_{n=0}^{\infty} \|Te_n\|^2 < \infty \Longrightarrow T \in \mathscr{K}(\mathscr{H}). \tag{4}$$

Problem 3 (20 points)

(a) Let $\mathbb{T}$ be the standard 1D torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. For which natural numbers k is $C^k(\mathbb{T})$ dense in $L^2(\mathbb{T})$ (that is, with respect to the norm on $L^2(\mathbb{T})$)? Justify your answer.

(b) Suppose $f \in L^2(\mathbb{T})$ has the property that $\exists M$ such that

$$\forall \varphi \in C^1(\mathbb{T}) \quad \left| \int_{\mathbb{T}} f\varphi' \right| \leq M\|\varphi\|_{L^2}.$$

Prove that there exists a function g such that (1) $g \in L^2(\mathbb{T})$, and (2)

$$\forall \varphi \in C^1(\mathbb{T}) \quad \int_{\mathbb{T}} g\varphi = -\int_{\mathbb{T}} f\varphi'.$$

Problem 4 (20 points) Let M be a linear subspace of a normed linear space X, and let $x_0 \in X$ with $\operatorname{dist}(x_0, M) = \delta > 0$ where $\operatorname{dist}(x, V) \stackrel{\text{def}}{=} \inf_{v\in V} \|x - v\|$. Prove that there is a bounded linear functional $\varphi \in X^*$ such that $\|\varphi\| \leq \delta^{-1}$, $\varphi(x_0) = 1$, and $\varphi(m) = 0 \ \forall m \in M$.

Problem 5 (20 points)

Let $\Omega = [0, 1]$ and let the kernel $\kappa : \Omega \times \Omega \to \mathbb{C}$ be a symmetric function, meaning that $\kappa(x, y) = \overline{\kappa(y, x)}$.

Assume that κ is *positive-definite* meaning that for all $n \in \mathbb{N}$ and all complex numbers $(c_1, \ldots, c_n)$ and points $(x_1, \ldots, x_n) \subset \Omega$, then

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{c_i} c_j \kappa(x_i, x_j) \geq 0.$$

In other words, the matrix $K^{(n)}$ with entries $K^{(n)}_{i,j} = \kappa(x_i, x_j)$ is a Hermitian positive semi-definite matrix. *Note: you may assume all fields are real rather than complex if you wish.*

We assume κ is (jointly) continuous and not identically equal to 0. Let $\mathscr{H} \stackrel{\text{def}}{=} L^2(\Omega)$. Associated with κ is the linear operator $\mathcal{T}_\kappa : \mathscr{H} \to \mathscr{H}$ defined by $\left(\mathcal{T}_\kappa f\right)(x) = \int_\Omega \kappa(x, y)f(y)\,dy$. We'll prove parts of the classic *Mercer's Theorem*.

(a) Prove that $\mathcal{T}_\kappa$ has at least one nonzero eigenvalue.

(b) Prove that all eigenvalues of $\mathcal{T}_\kappa$ are nonnegative.