Applied Analysis Preliminary Exam

9:00-12:00 January 6, 2022

Instructions: You have three hours to complete this exam. Work all five problems; each is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are being asked to prove such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name.

Problem 1:

(a) Show that

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{ke^{k/n}}{n^2}$$

exists. For extra credit, what is A?

(b) Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f_1(x) = e^{x_1}\cos(x_2)$, $f_2(x) = e^{x_1}\sin(x_2)$ where $x = (x_1, x_2)$. Use the Inverse Function theorem to show that f is locally invertible.

(c) Is f given in (b) globally invertible? Explain.

(d) Suppose that $g : \mathbb{R} \to \mathbb{R}$ is differentiable and $g'(x) \neq 0$ for all $x \in \mathbb{R}$. Show that g is globally invertible on its range.

Problem 2:

(a) Consider the nonlinear integral equation

$$f(x) - \frac{1}{10}\int_0^1 (x + y^2) f^2(y)\, dy = \frac{1}{3}.$$

Show that there is a unique continuous solution $f : [0, 1] \to \mathbb{R}$ of this equation with the property that $0 \le f(x) \le 1$ for all $x \in [0, 1]$.

(b) Consider the function $f : \mathbb{R}^6 \to \mathbb{R}^2$, with variables (u, v, w, x, y, z) defined by

$$f_1 = u^2 + v^2 + w^2,$$
$$f_2 = xu^2 - yv^2 + zw^2.$$

(1) Find Df.

(2) Consider the point $(1, 1, 1, 2, 1, -1) \in \mathbb{R}^6$. Let $f(1, 1, 1, 2, 1, -1) = f_0$. Show that there exist two functions

$$u : \mathbb{R}^4 \to \mathbb{R}, \quad v : \mathbb{R}^4 \to \mathbb{R},$$

of the four variables (w, x, y, z) that are continuously differentiable on some ball B centered at the point $(w, x, y, z) = (1, 2, 1, -1)$, such that $u(1, 2, 1, -1) = 1$, $v(1, 2, 1, -1) = 1$, and the equations $f(u, v, w, x, y, z) - f_0 = 0$ both hold for all $(w, x, y, z) \in B$.

(3) Can the implicit function theorem be applied at the same point to find functions (v, w) of the variables (u, x, y, z) that satisfy $f - f_0 = 0$? Why or why not?

Problem 3: Let X and Y be Hilbert spaces and $T : X \to Y$ be a bounded linear operator. Prove that if T is compact then its adjoint $T^* : Y^* \to X^*$ is also compact.

Problem 4: Let D be a countable set. Prove that any sequence of functions $f_n : D \to E$ such that the set $\{f_n(d)\}_{n=1}^{\infty}$ is precompact for each $d \in D$ has a subsequence which is point-wise convergent in D.

Problem 5: Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function. Prove that f maps sets of Lebesgue measure zero to sets of Lebesgue measure zero.