Solutions: APPM 3310-001: Matrix Methods — Final Exam — Fall 2011

1. (30 points) Consider the set of equations $-3x+4y-6z=-5/2$, $\quad 6x-8y+12z=c$, where c is some real number.

   (a) Write the system in matrix form $A\mathbf{x}=\mathbf{b}$.

   (b) Define the fundamental subspaces **range** and **cokernel** for an arbitrary matrix. Find these spaces for A.

   (c) Define the **Fredholm compatibility conditions** (Fredholm alternative) for a general system $A\mathbf{x}=\mathbf{b}$. Find a value of c that satisfies these conditions for the system in (a).

   (d) For this c value, find the general solution to (a). Write the solution as $\mathbf{x}=\mathbf{w}+\mathbf{z}$, where $\mathbf{w}\in \operatorname{corng}(A)$ and $\mathbf{z}\in\ker(A)$.

   (e) Of the solutions in (d), which has the smallest Euclidean norm?

Solutions:

(a) $[A\mathbf{x}|\mathbf{b}] = \left[\begin{array}{ccc|c} -3 & 4 & -6 & -5/2 \\ 6 & -8 & 12 & c \end{array}\right]$

(b) **range**$(A) = \{\mathbf{b}|\mathbf{b}=A\mathbf{x}$ for some $\mathbf{x}\}$ and **cokernel**$(A) = \{\mathbf{y}|A^T\mathbf{y}=0\}$

Note here,

$$\left[\begin{array}{ccc|c} -3 & 4 & -6 & b_1 \\ 6 & -8 & 12 & b_2 \end{array}\right] \to \left[\begin{array}{ccc|c} -3 & 4 & -6 & b_1 \\ 0 & 0 & 0 & 2b_1+b_2 \end{array}\right]$$

so $\operatorname{rng}(A) = \operatorname{span}\left\{\begin{pmatrix} -3 \\ 6\end{pmatrix}\right\} = \left\{\mathbf{b}\in\mathbb{R}^2 \middle| \begin{pmatrix} b_1 \\ b_2\end{pmatrix}\cdot\begin{pmatrix} 2 \\ 1\end{pmatrix}=0\right\}$ and since the range is orthogonal to the cokernel we have $\operatorname{coker}(A) = \operatorname{span}\left\{\begin{pmatrix} 2 \\ 1\end{pmatrix}\right\}$

(c) <u>Fredholm:</u> The linear system $A\mathbf{x}=\mathbf{b}$ has a solution if and only if $\mathbf{b}$ is orthogonal to the cokernel of A.

From part (b) with $\mathbf{b}=\begin{pmatrix} -5/2 \\ c\end{pmatrix}$, we need $\begin{pmatrix} -5/2 \\ c\end{pmatrix}\cdot\begin{pmatrix} 2 \\ 1\end{pmatrix}=0$, that is, $-5+c=0$, so we need $c=5$ for consistency.

(d) Note ,

$$\left[\begin{array}{ccc|c} -3 & 4 & -6 & -\frac{5}{2} \\ 6 & -8 & 12 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} -3 & 4 & -6 & -\frac{5}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{array}{rcl} -3x+4y-6z &=& -\frac{5}{2} \\ y &=& y \\ z &=& z \end{array} \to \begin{array}{rcl} x &=& 5/6+4/3y-2z \\ y &=& y \\ z &=& z \end{array}$$

so a general solution is $\mathbf{x} = \begin{pmatrix} 5/6 \\ 0 \\ 0\end{pmatrix} + \begin{pmatrix} 4/3 \\ 1 \\ 0\end{pmatrix}t + \begin{pmatrix} -2 \\ 0 \\ 1\end{pmatrix}s = \begin{pmatrix} 5/6+4/3t-2s \\ t \\ s\end{pmatrix}$ Now to find a solution in the corange, we need

$$\begin{pmatrix} 5/6+4/3t-2s \\ t \\ s\end{pmatrix}\cdot\begin{pmatrix} 4/3 \\ 1 \\ 0\end{pmatrix}=0 \text{ and } \begin{pmatrix} 5/6+4/3t-2s \\ t \\ s\end{pmatrix}\cdot\begin{pmatrix} -2 \\ 0 \\ 1\end{pmatrix}=0$$

since the corange is orthogonal to the kernel. So we get

$$\begin{array}{r} 10/9+16/9t-8/3s+t=0 \\ -5/3-8/3t+4s+s=0 \end{array} \to \begin{array}{r} 25/9t-8/3s=-10/9 \\ -8/3t+5s=5/3 \end{array} \to \begin{array}{r} t=-10/61 \\ s=15/61 \end{array}$$

so the solution in the corange is $\mathbf{w} = \begin{pmatrix} 15/122 \\ -10/61 \\ 15/61 \end{pmatrix}$, and so we have,

$$\mathbf{x} = \mathbf{w} + \mathbf{z} = \begin{pmatrix} 15/122 \\ -10/61 \\ 15/61 \end{pmatrix} + \begin{pmatrix} 4/3 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} s, \ \forall t, s \in \mathbb{R}$$

(e) The solution in the corange, $\mathbf{w} = \begin{pmatrix} 15/122 \\ -10/61 \\ 15/61 \end{pmatrix}$, has the smallest norm.

2. (30 points) **State the fundamental theorem of linear algebra**, then, for each property given below, write down a matrix F with that property or explain why no such matrix exists.

   (a) $\operatorname{rng}(F) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$, and $\operatorname{corng}(F) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$.

   (b) $F = F^{-1}$.

   (c) The vector $\begin{pmatrix} 10 \\ 6 \\ 90 \end{pmatrix}$ is in the kernel of F, $\begin{pmatrix} -6 \\ 10 \\ 0 \end{pmatrix}$ is in the corange of F, and $\det F = 1$.

   (d) F is real and symmetric with eigenvalues $1 + i$ and $1 - i$.

   (e) F has an eigenvalue 2 with multiplicity two, but only one eigenvector.

Solutions:

Fund. Thm. of Linear Algebra:

Let A be an $m \times n$ matrix or rank r. Then

$$\dim(\operatorname{corng}(A)) = \dim(\operatorname{rng}(A)) = \operatorname{rank}(A) = \operatorname{rank}(A^T) = r$$

and $\dim(\ker A) = n - r$ and $\dim(\operatorname{coker}(A)) = m - r$.

(a) Here any matrix of the from $F = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ where $b \neq 0$ will suffice.

(b) Let $F = I$.

(c) Not possible, if the kernel is nontrivial then $\lambda = 0$ is an eigenvalue of the matrix and so $\det F = 0$ since the determinant is the product of the eigenvalues.

(d) Not possible, if F is real and symmetric then F has to have real eigenvalues.

(e) Let $F = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, then 2 is a repeated eigenvalue but there will be only one free variable in the eigenvalue equation and thus only one eigenvector.

3. (30 points) Let $q(x, y, z) = x^2 + 2xy + 3y^2 + 4yz + 4z^2$ be a quadratic form.

   (a) Write $q = \mathbf{x}^T K \mathbf{x}$ for a matrix K.

   (b) Compute the LU decomposition of K.

   (c) Define what it means to be a **positive definite matrix**. Is q a positive definite quadratic form? Explain.

   (d) What do (a)-(c) allow you to conclude about the eigenvalues of K? (**DO NOT COMPUTE** the λ's)

(e) $\lambda_1 = 2$ is an eigenvalue of K. Find its associated eigenvector.

(f) What is the sum of the other two eigenvalues, $\lambda_2 + \lambda_3$? (Hint: You **DO NOT** compute λ_2 and λ_3 separately.)

Solutions:

(a) $q = \mathbf{x}^T \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 2 & 4 \end{pmatrix} \mathbf{x}$

(b) $L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$

(c) K is positive definite if it is symmetric and if $\mathbf{x}^T K \mathbf{x} > 0$ for all $\mathbf{x} \neq 0$.

Yes, since $q = \mathbf{x}^T K \mathbf{x}$, then $q > 0$ for all $\mathbf{x} \neq 0$, and so q will be a positive definite quadratic form by definition.

(d) The eigenvalues of K must be real and positive since the eigenvalues of a positive definite matrix are positive. We know from (a) that K is symmetric and we know from (b) that K is regular with positive pivots and therefore is positive definite. The eigenvalues are also complete since K is symmetric and real. Finally, note that the eigenvalues can't all be the same since $\det K = 4$ and $\operatorname{tr}(K) = 8$

(e) The associated eigenvector is $\mathbf{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

(f) The sum of the other two eigenvalues is 6 since the sum of the eigenvalues is the trace of the matrix, which is 8, and so if one eigenvalue is 2, the sum of the other two eigenvalues must be 6.

4. (20 points) Let $S = \operatorname{span}\{(x-1), (x-1)^2\}$.

(a) Define **vector subspace**.

(b) Show that S is a vector subspace of $P^{(2)}$, the space of quadratic polynomials.

(c) What is $\dim(S)$?

(d) Define the L^2 **inner product** on the interval $(0, 2)$.

(e) Find the orthogonal complement $S^\perp$ to S in $P^{(2)}$ using the inner product in (d).

Solutions:

(a) A vector subspace is a subset of a vector space that is closed under the addition and scalar multiplication of the vector space.

(b) In $P^{(2)}$, the addition and scalar multiplication are the usual additional and scalar multiplication of polynomials and the set $\operatorname{span}\{(x-1), (x-1)^2\}$ is, by definition, closed under addition and scalar multiplication and so S is a subspace of $P^{(2)}$.

(c) Note that $c_1(x-1) + c_2(x-1)^2 = 0$ implies $c_1 = c_2 = 0$ and so the vectors $p_1 = (x-1)$ and $p_2 = (x-1)^2$ form a basis of S and thus $\dim S = 2$.

(d) For any $f(x),\ g(x) \in C((0,2))$ we have $\langle f, g \rangle = \displaystyle\int_0^2 f(x)g(x)dx$

(e) Note that $S^\perp = \left\{ f \in P^{(2)} \middle| \langle f, p \rangle = 0,\ \forall p \in S \right\}$ and so,

$$S^\perp = \operatorname{span}\left\{ x^2 - 2x - \frac{1}{5} \right\} = \operatorname{span}\{5x^2 - 10x - 1\}$$

5. (20 points) Let $L[\mathbf{x}] = \begin{pmatrix} x_1 + 3x_2 \\ -x_1 + x_2 \\ 2x_2 \end{pmatrix}$ be a linear transformation.

(a) Which vector space is the domain of L? Which vector space is the co-domain of L?

(b) Suppose $\{\mathbf{v}_i : i = 1, \ldots n\}$ is a basis for the domain of L. What is n? Why? (DO NOT FIND $\mathbf{v}_i$.)

(c) If $L[\mathbf{v}_1] = \begin{pmatrix} 7 \\ 1 \\ 4 \end{pmatrix}$, what is $\mathbf{v}_1$?

(d) Find a basis for the range of L.

Solutions:

(a) Note, $L \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$

(b) $n = 2$ since the domain of L is $\mathbb{R}^2$ and $\dim(\mathbb{R}^2) = 2$.

(c) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(d) $\text{rng}(L) = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right\}$

6. (20 points) Let Q be the matrix

$$Q = I - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}$$

where $\mathbf{v}$ is an n-dimensional vector. This type of matrix is called a Householder reflection matrix.

(a) What is the size of the matrix $\mathbf{v}^T\mathbf{v}$? Of the matrix $\mathbf{v}\mathbf{v}^T$?

(b) Suppose the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Show that $Q^TQ = I$.

(c) For an arbitrary $\mathbf{v}$, show that $Q = Q^T$.

(d) Show that $Q^TQ = I$ for any arbitrary $\mathbf{v}$.

Solutions:

(a) $\mathbf{v}^T\mathbf{v}$ is a scalar (1×1) and $\mathbf{v}\mathbf{v}^T$ is an $n \times n$ matrix.

(b) Note $\mathbf{v}^T\mathbf{v} = 2$ and so

$$Q = I - \frac{2}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and finally, } Q^TQ = I$$

(c) Note that since $\mathbf{v}^T\mathbf{v}$ is a scalar we have $Q = I - \frac{2}{\mathbf{v}^T\mathbf{v}}\mathbf{v}\mathbf{v}^T$ and so,

$$Q^T = \left(I - \frac{2}{\mathbf{v}^T\mathbf{v}}\mathbf{v}\mathbf{v}^T\right)^T = I^T - \frac{2}{\mathbf{v}^T\mathbf{v}}(\mathbf{v}\mathbf{v}^T)^T = I - \frac{2}{\mathbf{v}^T\mathbf{v}}(\mathbf{v}^T)^T\mathbf{v}^T = I - \frac{2}{\mathbf{v}^T\mathbf{v}}\mathbf{v}\mathbf{v}^T = Q$$

(d) Using part (c) $Q^TQ = Q^2$ and now,

$$\begin{aligned} Q^2 = \left(I - \frac{2}{\mathbf{v}^T\mathbf{v}}\mathbf{v}\mathbf{v}^T\right)\left(I - \frac{2}{\mathbf{v}^T\mathbf{v}}\mathbf{v}\mathbf{v}^T\right) &= I - \frac{4}{\mathbf{v}^T\mathbf{v}}\mathbf{v}\mathbf{v}^T + \frac{4}{(\mathbf{v}^T\mathbf{v})^2}\mathbf{v}\underbrace{\mathbf{v}^T\mathbf{v}}_{\text{scalar}}\mathbf{v}^T \\ &= I - \frac{4}{\mathbf{v}^T\mathbf{v}}\mathbf{v}\mathbf{v}^T + \frac{4}{(\mathbf{v}^T\mathbf{v})^2}(\mathbf{v}^T\mathbf{v})\mathbf{v}\mathbf{v}^T \\ &= I - \frac{4}{\mathbf{v}^T\mathbf{v}}\mathbf{v}\mathbf{v}^T + \frac{4}{\mathbf{v}^T\mathbf{v}}\mathbf{v}\mathbf{v}^T = I \end{aligned}$$

♠ END ♡