APPM 2360 Project 1

Due: Friday October 6 **BEFORE 5 P.M.**

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1 Introduction

A pair of close friends are currently on the market to buy a house in Boulder. Both have obtained engineering degrees from CU and have an understanding of differential equations but lack the skills necessary to decide what type of mortgage to take out on their new house. In the following sections, you'll find some research they have done on commonly used mortgage structures. However, they have become busy and cannot continue their analysis. Since you are in Differential Equations this semester, they have decided to enlist your help in writing a report to understand their options.
Your task is to read the following sections and complete the tasks found in Section 3. Your friends were nice enough to list various problems they would like solved. However, you should be careful to write your responses in a cohesive report.

2 Background Information

In this section you will find information on some of the most prevalent mortgage structures.

2.1 Interest Compounding

Often, the interest of a loan is expressed as the annual rate, that is, the percentage of the outstanding balance that is charged as interest over a year. However, the frequency with which the rate is applied to the current balance may vary. This frequency is how often the loan *compounds*. If the interest is compounded annually, the formula to calculate the amount of money owed after the first year is

$$y(1) = (1 + r)y(0),$$

where $y(t)$ is the outstanding balance after t years, and r is the annual interest rate expressed as a fraction (e.g., an interest rate of 1% corresponds to $r = 0.01$).
How would this change if, instead, the loan compounded semiannually? Then, half the interest rate would be applied to the loan value every 6 months.

$$y(0.5) = \left(1 + \frac{r}{2}\right) y(0),$$

$$y(1) = \left(1 + \frac{r}{2}\right) y(0.5) = \left(1 + \frac{r}{2}\right)\left(1 + \frac{r}{2}\right) y(0) = \left(1 + \frac{r}{2}\right)^2 y(0).$$

This pattern continues for any frequency of compounding. That is, if a loan is compounded n times per year, the value of the loan after 1 year is

$$y(1) = \left(1 + \frac{r}{n}\right)^n y(0).$$

More generally, we can write the expression at time t for a loan compounded n times annually as

$$y(t) = \left(1 + \frac{r}{n}\right)^{nt} y(0). \tag{1}$$

The more frequently that a loan compounds, the higher the value at the end of the year. However, there is a limit as n goes to infinity. The limit, which models a *continuously compounding* loan, is:

$$y(t) = y(0)e^{rt}.$$

This model can also be expressed as a differential equation:

$$\begin{aligned} y' &= ry, \\ y(0) &= y_0. \end{aligned}$$

If the borrower makes a monthly payment of p dollars, the model becomes, assuming that the payments are distributed continuously throughout the year,

$$\begin{aligned} y' &= ry - 12p, \\ y(0) &= y_0. \end{aligned} \tag{2}$$

2.2 Adjustable vs. fixed rate mortgages

Some mortgages use a fixed annual rate, i.e., the value of r in the model is constant. Fixed rate loans are sold with a minimum monthly payment, which ensures that the loan will be paid off within the duration. The bank will get all of the money back sooner with a shorter duration, so the rate is lower for this type of loan.
Adjustable rate mortgages start at a lower rate than a fixed rate loan, but after a certain period of time, the interest rate becomes tied to one of several public indexes. This results in the possibility of the interest rate increasing above the rate that a fixed-rate mortgage would have for the second portion of the loan. Common periods during which the interest rate is fixed are 3,5,7, or 10 years. Mathematically, this results in r being a function of time $r(t)$ instead of a constant, and the model is

$$\begin{aligned} y' &= r(t)y - 12p, \\ y(0) &= y_0. \end{aligned} \tag{3}$$

3 Questions

Your friends recently found a house on the market and needed to borrow **\$750,000** from the bank in order to purchase. Your friends are very careful when reading mathematics so in your analysis you should include as much relevant work as necessary so your friends can follow. They provided you a list of questions that they need answered.

3.1 Analysis of Fixed Rate Mortgages

Your friends have various options when choosing a mortgage structure. The following points should help you know what to include in your analysis.

1. Examine the effect of continuous compounding on the value of a loan. Assuming that $r = 0.03$ and the original balance is $\$750,000$, compute the total cost of the loan after 5 years for loans compounded $1, 2, 4$, and 12 times per year, without any payments with equation (1). Use equation (2) to compare these values to the value of the loan compounded continuously. Plot the value of the loan compounded 4 times a year and 12 times a year as well as the value of the loan when the interest is compounded continuously as a function of time.

2. Next, gain a broad understanding of the behavior of the loan value by determining whether there any equilibrium solutions to (2). If so, what are they, and what is their stability? What do these equilibria represent in real-word terms?

3. Determine the exact behavior of the loan in your friends' situation by solving (2) using separation of variables, with $y(0) = y_0$ and r and p arbitrary.

4. The size of the monthly payment p that your friends are willing to make plays a large role in deciding the type of loan they should choose. Use the solution to (2) to find the correct p to pay off a 10 year fixed rate mortgage with rate 3% and initial debt of $750,000$. Do the same for a 30 year fixed-rate mortgage with rate 5%. ***Hint: you want to find*** t ***such that*** $y(t) = 0$***.***

5. While having a low monthly payment is nice, you should warn your friends that there is quite literally a price to pay for this convenience. We can determine the total paid by summing each monthly payment over the duration of the loan. How much interest is paid in the 30 year fixed rate mortgage? The 10 year?

6. Buyers often choose to pay as much of the cost as they can up front so that they don't have to borrow quite so much. Might this option be worth it for your friends? How much money would the borrower save in each case if he/she paid \$100,000 down on the house? (That is, the mortgage began at \$650,000.)

7. What are the advantages and disadvantages of taking out a 30 year fixed rate mortgage as opposed to a 10 year mortgage?

3.2 Numerical solutions

Often, the differential equations we wish to solve will be difficult or impossible to solve by hand so we enlist the help of a numerical scheme. Here we will use Euler's method. First we will use it on Eq. (1) in order to compare its results with the known analytical solution you found in item 3 of Section 3.1. Once we are satisfied that the method is working in this known case, we will use it in the case where the interest rate is variable.

3.2.1 Fixed rate mortgage

Consider a mortgage for \$750,000 with a constant interest rate of 5% ($r = 0.05$) and a monthly payment $p = \$4,000$.

1. Implement Euler's method for Eq. (2) with step size $h = 0.01$. Use a *while* loop to run the method until the mortgage is paid off ($y = 0$) and determine when it is paid off.

2. Plot the numerical solution $y(t)$ and the true solution to equation (2) with the parameters given here in the same graph and compare the two.

3. Repeat the previous item for a step size $h = 0.5$ and comment on the difference.

3.2.2 Adjustable rate mortgage

Now we turn to the adjustable rate mortgages. Suppose that for the same \$750,000 mortgage a bank offers an adjustable rate mortgage, which starts with an initial lower fixed rate of 3% ($r = 0.03$) for the first 5 years and is tied to credit markets after that. Let's assume that after the first 5 years the rate increases as $r(t) = 0.03 + 0.015\sqrt{t-5}$, so

$$r(t) = \begin{cases} 0.03 & t \le 5, \\ 0.03 + 0.015\sqrt{t-5} & t > 5. \end{cases} \tag{4}$$

1. Suppose the borrower pays \$4000 per month. How long will it take him/her to pay off the mortgage?

2. What about if he/she pays \$4500 per month?

3. How much interest is paid in each case?

4. Plot the numerical solution $y(t)$ for both scenarios on the same graph. Explain what is going on in it. How does the interest rate affect the graph?

3.3 Conclusion

Write a brief letter to your friends. In this letter you must address the following items

- The type of mortgage you recommend to your friends.
- The advantages/disadvantages of having a longer mortgage vs. a shorter mortgage.

4 Items to Remember

- Your friends want a complete report so you should write full sentences explaining the questions posed and your responses. Don't simply number your responses to individual questions.

- All reports must be submitted to D2L by the due date in a .pdf format. Failure to do so will result in a penalty. Any `MATLAB` code must also be submitted to D2L.
- `MATLAB` code is bulky and hard to read. If you would like to include it in your report, do so in an appendix.
- Your friends are sticklers for the rules, so they really want you to follow all of the project guidelines on the APPM 2360 webpage. Be sure to read these carefully before starting your project!