This is a sample Lab assignment for Calc III. It is based on a Calc I style optimization problem, and this is intended to show examples of "good" and "bad" lab write-ups. This lab would be considered "good" and would likely result in a grade in the 'A' range. Note: Your lab should be double spaced, and have a title page. This is single spaced so it fits on two pages

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# 1 Introduction

King Ottokar II of Styria wants to build a cage for his new pet unicorn. He wants this cage to be rectangular, with one side of the cage decorated and covered in gold (gilded). It is well known that King Ottokar I wanted to save money. So he built a cage that was very inexpensive, and way too small for his unicorn. One day, this unicorn broke free, and trampled the King [1]. Ottokar II doesn't want to make the same mistake as his father.

A happy unicorn will have a cage with area 2500 square cubits. The three cheaper sides of the cage cost 3 ducats per running cubit, while the gilded side costs 51 ducats per running cubit.

In this lab, we will help Ottokar II find the optimum size of the cage. That is, the dimensions that will minimize the cost of the cage, but still have an area of 2500 square cubits.

# 2 Designing the Cage

The King wants to build a rectangular cage to display the unicorn at court. We will define this rectangular cage with two sides of length $x$ and two sides of length $y$. For a diagram of this cage see Figure 1. Everyone knows that happy unicorns require an area of 2500 square cubits. Since our cage's area is $xy$, we know that $xy = 2500$. The gilded wall costs 51 ducats per running cubit while the other three walls costs 3 ducats per running cubit. Because the cost of the gilded wall is much greater than the cost of other walls, the gilded wall and the wall opposite it should be shorter than the other two walls. Having a rectangular cage with a shorter gilded wall will help minimize the cost. The gilded wall will be one of the walls with length $y$.

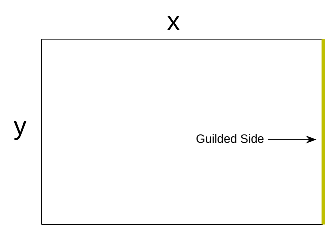


Figure 1: Diagram of the unicorn's cage.

The cost of the cage depends on how much of each material is used. Three of the walls cost 3 ducats per running cubit and the gilded wall costs 51 ducats per running cubit. The cost of the cage is $C(x, y) = 3x + 3x + 3y + 51y$, where the fourth wall is the gilded wall. We want to simplify this to an equation for the cost as a function of the longer side, $x$. Since we know that the area of the cage is $xy = 2500$, we can solve for $y$ as a function of $x$. Doing this gives $y = \frac{2500}{x}$. Replacing the $y$ in $C(x, y)$ with $y = \frac{2500}{x}$ and simplifying, gives the equation for the cost as a function of the longer side as $c(x) = 6x + \frac{135000}{x}$. To see how the cost varies as the length of side $x$ changes, see the plot of the cost function in Figure 2. From the figure, it is clear there is some value of $x$ that minimizes the cost.

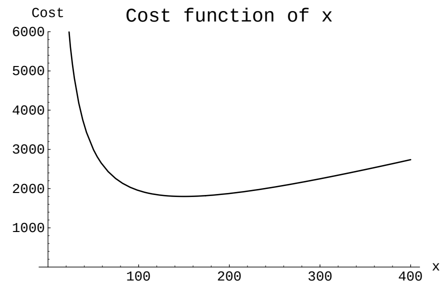


Figure 2: The cost of the cage as a function of the length of the longer side.

## 3 Determining the Cheapest Way to Build the Cage

We need to use calculus to determine the minimum value of the cost of the cage. In order to find where extreme values occur, we need to find where the derivative of the cost function is zero. The derivative of $c(x)$ is $c'(x) = 6 - \frac{135000}{x^2}$. Solving this for $x$ gives $x = 150$. To determine if this is a maximum or minimum, we need to look at the value of the second derivative at $x = 150$. The second derivative is $c''(x) = \frac{270000}{x^3}$ so $c''(150) = 0.08$. Since the second derivative is positive, the cost function is concave up at $x = 150$ so it is a minimum. We now calculate the minimum cost as $c(150) = 6(150) + \frac{135000}{150} = 1800$ ducats. We know two sides have lengths of 150 cubits. We can find the lengths of the other two sides using $y = \frac{2500}{x}$. Plugging in $x = 150$, gives the lengths of the other two sides as $\frac{50}{3} \approx 16.7$ cubits. Therefore, our happy unicorn lives in a cage that is 150 cubits by 16.7 cubits and costs 1800 ducats.

## 4 A Not So Fancy Cage

If we do away with the gilded wall to save on ducats (but making our unicorn a little less happy), all four walls will cost 3 ducats. This changes the cost function to $c(x) = 6x + \frac{6(2500)}{x}$. Again, we use calculus to find the minimum cost. The first derivative of the cost function is $c'(x) = 6 - \frac{15000}{x^2}$. Solving this for $x$ gives $x = 50$. A check of the second derivative shows that this is indeed a minimum. With all walls costing 3 ducats, the optimal cage is square with sides of length 50 cubits. The cage only costs 600 cubits.

## 5 Conclusions

We found that having a happy unicorn in a gilded cage requires a rectangular cage of size 150 cubits by 16.7 cubits. This cage costs 1800 ducats. If the King wants to be cheap and not have a gilded wall, the optimal cage for the happy unicorn is a square with sides of length 50 cubits. This non-gilded cage costs only 600 cubits.

In doing this lab, we learned a practical application of calculus — minimizing costs. If it wasn't for our help, King Ottokar II would have spent too much on his new cage.

## 6 References

[1] M. R. Calculus. A Brief History of Styria: From Ottocar I to Ottocar VIII. Sunnydale: Newton's Publishing Co., 2003.