Numerical Computations on the Painlevé Equations

Bengt Fornberg

University of Colorado at Boulder
Department of Applied Mathematics

in collaboration with:

André Weideman

University of Stellenbosch
Department of Mathematical Sciences

Jonah Reeger

Air Force Institute of Technology,
Wright Patterson Air Force Base
Department of Mathematics and Statistics

Painlevé Transcendents

Consider ODEs of the form $\frac{d^2u}{dz^2} = F(z,u,\frac{du}{dz})$ where *F* is a rational function of its arguments

Painlevé Property: The solution *u*(*z*) has no movable branch points in the complex plane. Out of 50 such equations, the solutions to 44 can be expressed as elementary or standard special functions.

The remaining six equations are

$$P_I \quad : \quad \frac{d^2u}{dz^2} = 6u^2 + z$$

$$P_{II} \quad : \quad \frac{d^2u}{dz^2} = 2u^3 + zu + \alpha$$

$$P_{III} \quad : \quad \frac{d^2u}{dz^2} = \frac{1}{u}\left(\frac{du}{dz}\right)^2 - \frac{1}{z}\frac{du}{dz} + \frac{\alpha u^2 + \beta}{z} + \gamma u^3 + \frac{\delta}{u}$$

$$P_{IV} \quad : \quad \frac{d^2u}{dz^2} = \frac{1}{2u}\left(\frac{du}{dz}\right)^2 + \frac{3}{2}u^3 + 4zu^2 + 2(z^2 - \alpha)u + \frac{\beta}{u}$$

$$P_V \quad : \quad \frac{d^2u}{dz^2} = \left(\frac{1}{2u} + \frac{1}{u-1}\right)\left(\frac{du}{dz}\right)^2 - \frac{1}{z}\frac{du}{dz} + \frac{(u-1)^2}{z^2}(\alpha u + \frac{\beta}{u}) + \gamma\frac{u}{z} + \frac{\delta u(u+1)}{u-1}$$

$$P_{VI} \quad : \quad \frac{d^2u}{dz^2} = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-z}\right)\left(\frac{du}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{u-z}\right)\frac{du}{dz} + \frac{u(u-1)(u-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{u^2} + \frac{\gamma(z-1)}{(u-1)^2} + \frac{\delta z(z-1)}{(u-z)^2}\right)$$

Paul Painlevé 1863 - 1933

French mathematician and politician

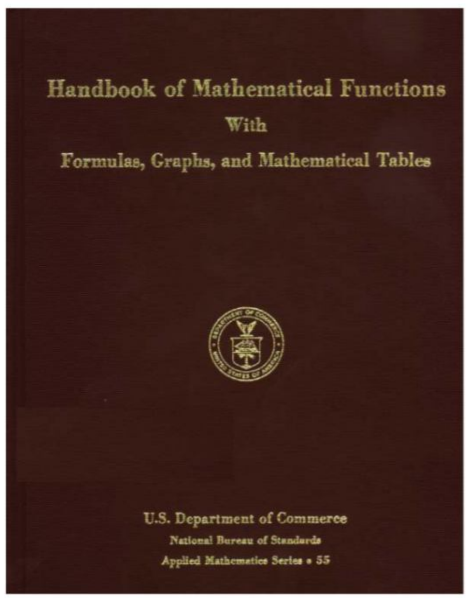
1917 and 1925 – 1929	Minister of war
1917 and 1925	Prime Minister of France
1932 – 1933	Minister of Air

Significance of the Painlevé equations

- The Painlevé equations arise as reductions of equations that are solvable by inverse scattering
- Intriguing link with the Riemann Hypothesis
- Statistical mechanics (Ising models)
- Random matrices; combinatorics
- Plasma physics
- Nonlinear waves (resonances in shallow water)
- Quantum gravity; quantum field theory
- General relativity; string theory
- Bose-Einstein condensates
- Raman scattering
- Nonlinear optics / Fiber optics

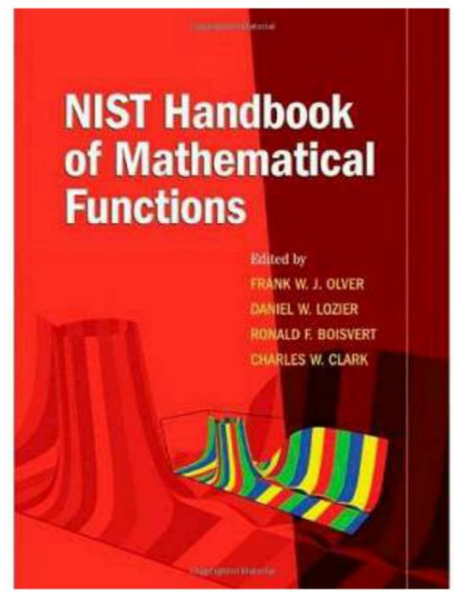
Abramowitz and Stegun (1964)

No mention of Painlevé equations



NIST Handbook of Mathematical Functions (2010)

One full chapter devoted to Painleve equations



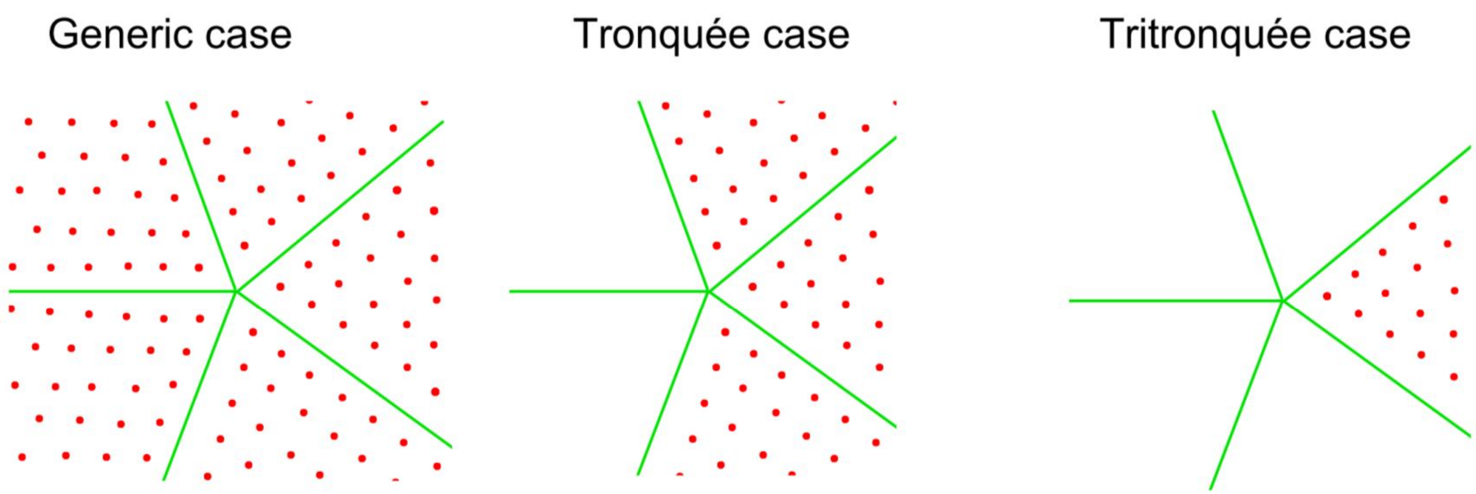
Consider first the P_I equation

Some analytic observations:

P_I : $$\frac{d^2u}{dz^2} = 6u^2 + z$$

Two parameter solution space since second order ODE
No additional parameters in the equation

Schematic pole field structures:



Most cases of $u(0)$ and $u'(0)$

Given $u(0)$, some choices of $u'(0)$

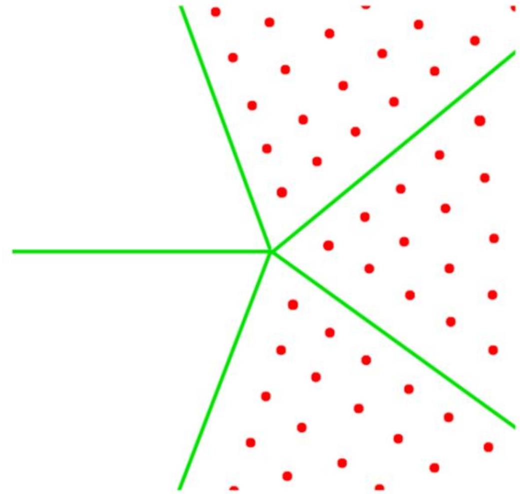
$u(0)$ = -0.1875543083404949...
$u'(0)$ = 0.3049055602612289...

- All poles are double, strength one and residue zero
- Asymptotically far out: Order 5 symmetry, since $u \to \zeta^3 u,\ z \to \zeta z$ with $\zeta^5 = 1$ leaves the ODE invariant
- In smooth sectors: $u(z) \approx \pm\sqrt{-z/6} + o(1)$
- No closed form solutions known

Some general numerical observations on solving the *P*-equations

Common (mis)perceptions:

- Pole fields: Numerical 'mine fields'
- Smooth sectors: Numerically easy



In reality:

- Pole fields well conditioned as initial value problems (IVPs)

  However, one needs a numerical scheme that does not degrade if a few poles are present locally

- In smooth sectors: Calculations ill conditioned as IVPs, but well conditioned as BVPs

  The ill conditioning apparent from dominant balance: $$\underbrace{\frac{d^2u}{dz^2}}_{\text{very small}} = \underbrace{6u^2 + z}_{\text{dominant balance}}$$

The two main components in the present numerical technique

(i) Utilize a 'pole-friendly' ODE initial value solver

General form of an ODE IVP: $y'(t) = f(t, y(t))$, Initial Condition (IC): $y(t_0) = y_0$.

Most basic numerical technique for ODE IVPs:

Forward Euler: $y(t+h) = y(t) + h\, f(t, y(t)) \quad (+O(h^2))$

Can view as first two terms of a Taylor expansion: first order accurate method

Three main ways to improve the order / numerical efficiency of Forward Euler:

- Runge-Kutta methods
- Linear multistep methods
- **Taylor expansion methods**
  - A very bad implementation Taylor expansion strategy is commonly described in numerical text books; highly effective versions are available
  - By an extra Padé (or continued fraction) step, one can obtain a numerical method that is perfectly suited for dense pole fields (Willers, 1974)

Taylor method in slightly more detail:

ODE: $y'(z) = f(z, y(z))$;

Taylor's method: $$y(z_n + h) = \underbrace{c_0}_{y(z_n)} + \underbrace{c_1}_{y'(z_n)} h^1 + c_2 h^2 + c_3 h^3 + \ldots$$

Steps:
- Obtain c_0 from current solution at z_n ,
- Then recursively substitute the truncated expansion into the ODEs RHS and integrate; gain one coefficient each 'time around'

Cost-effective to run out at each step to accuracy orders in range $m = 30 – 60$.

Padé conversion:

Convert to rational form, using the same degree $m/2$ in numerator and denominator

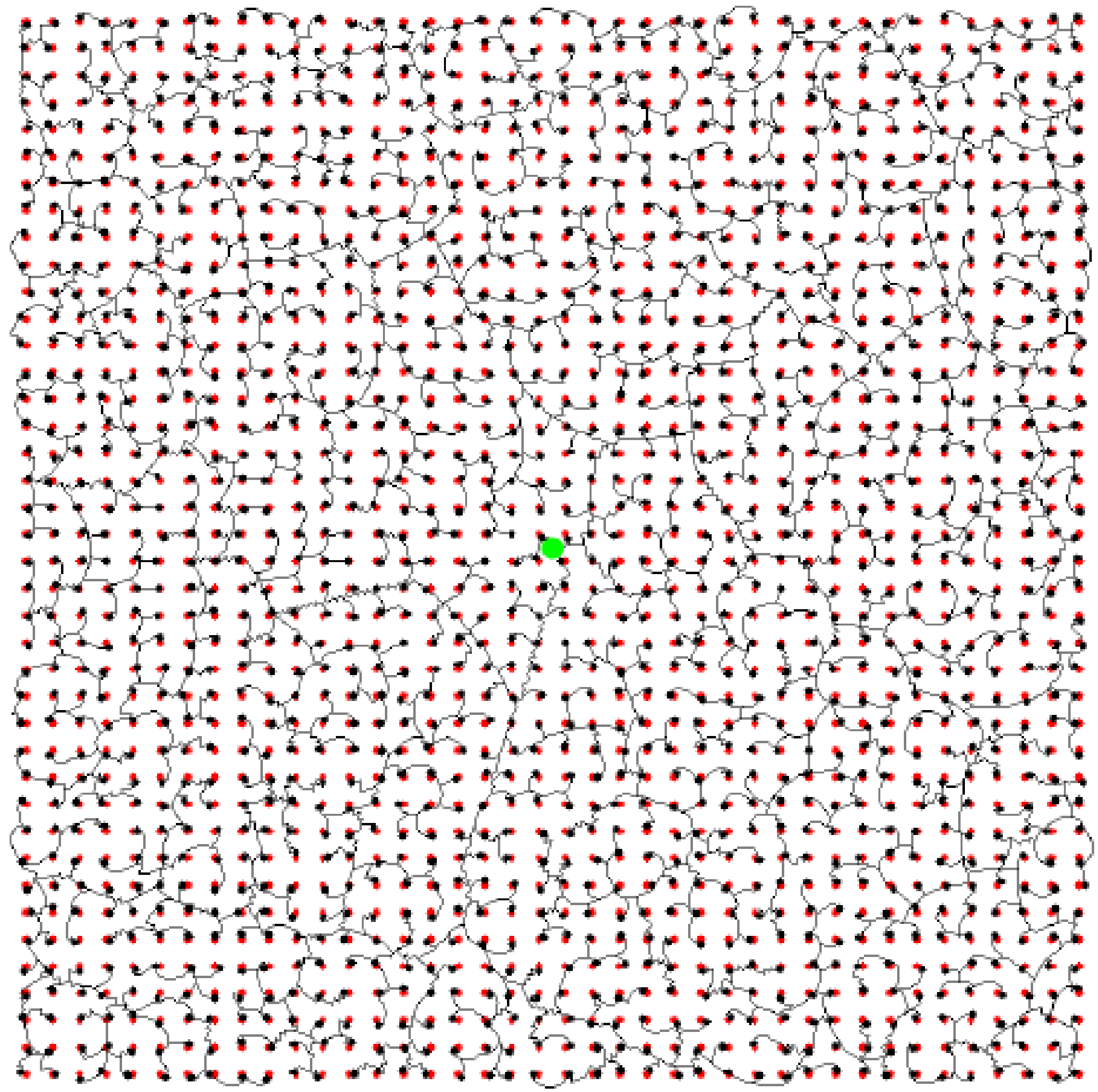
$$y(z_0 + h) = \frac{a_0 + a_1 h + a_2 h^2 + \ldots + a_{m/2} h^{m/2}}{1 + b_1 h + b_2 h^2 + \ldots + b_{m/2} h^{m/2}}$$

Still the same formal order, but a pole becomes now just a zero in the denominator; the functional form does not any longer limit where expression can be evaluated.

(ii) Path selection strategy across the complex plane

Stage 1:

Choose start point with given IC.
Select in random order lattice-based 'target points'
Run path to selected point from closest location so far; choose step in generally right direction, but keeping solution low.
In figure: 1,600 target points, 4,300 steps, 0.3 sec on typical notebook (in Matlab)



Stage 2:

Superpose much finer grid; fill in points with single Padé expansion from each end point of previous paths. Typically 0.4 sec.; total time around 0.7 sec.

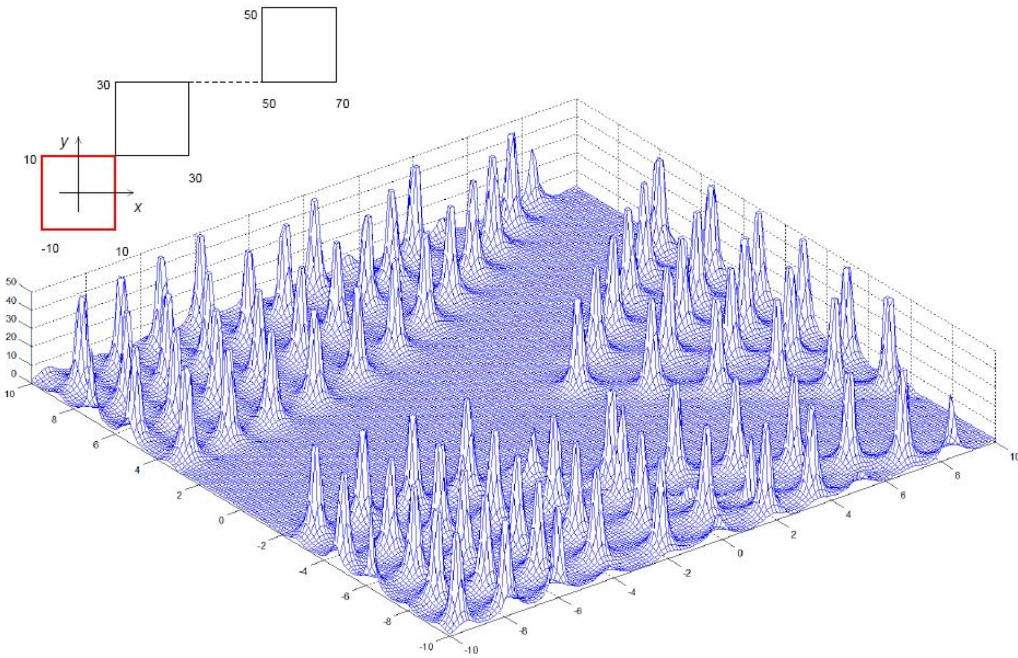
Example of solution fields: $u(0) = -0.1875, u'(0) = 0.3049$

IC near the tritronquée case:

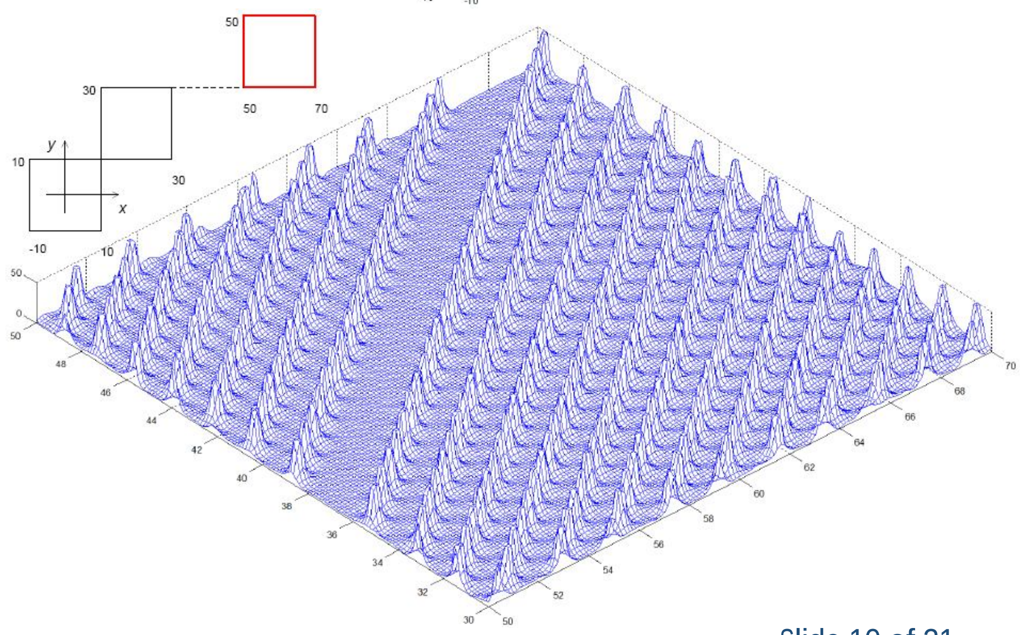
$u\,(0) = -0.1875543083404949$

$u'(0) = \;0.3049055602612289$

Magnitude of $u(z)$ displayed.



Within pole fields, accuracy typically better than 10^{-10} even at distances around 10^4.



NIST Handbook example:
$u(0) = 0$, $u'(0) = 1.85185403375822$

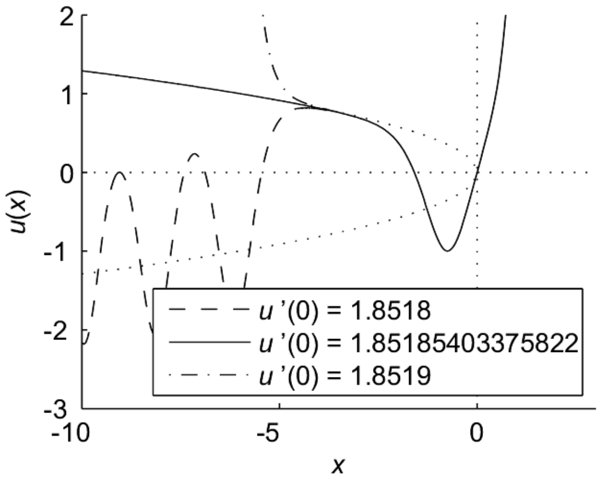
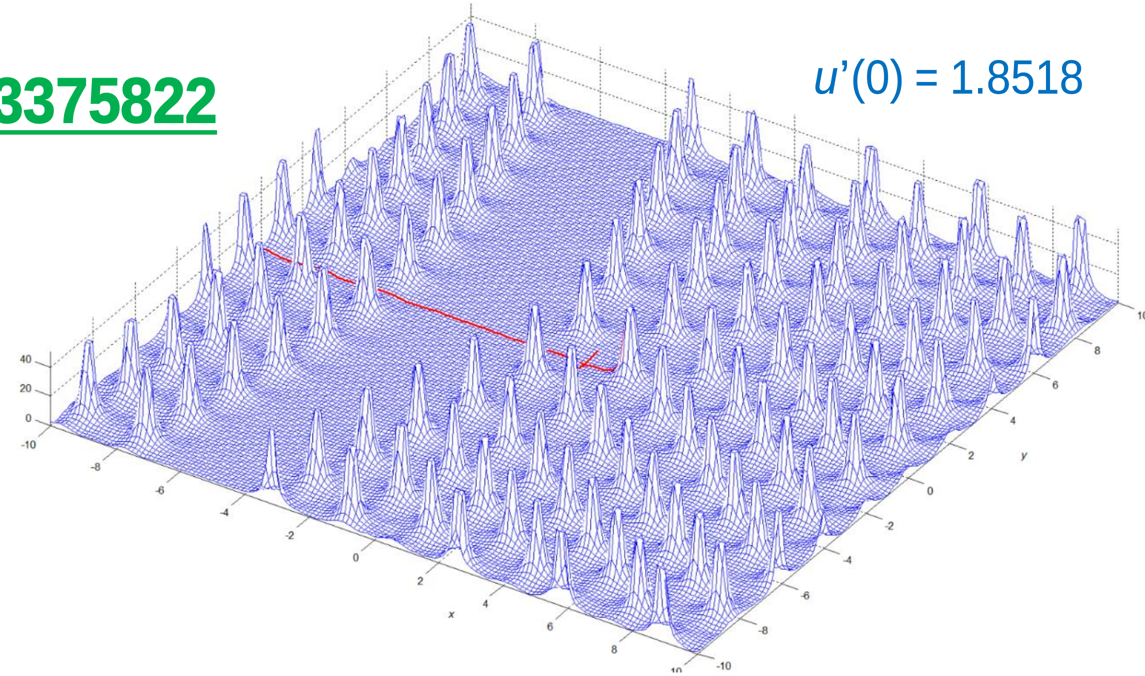


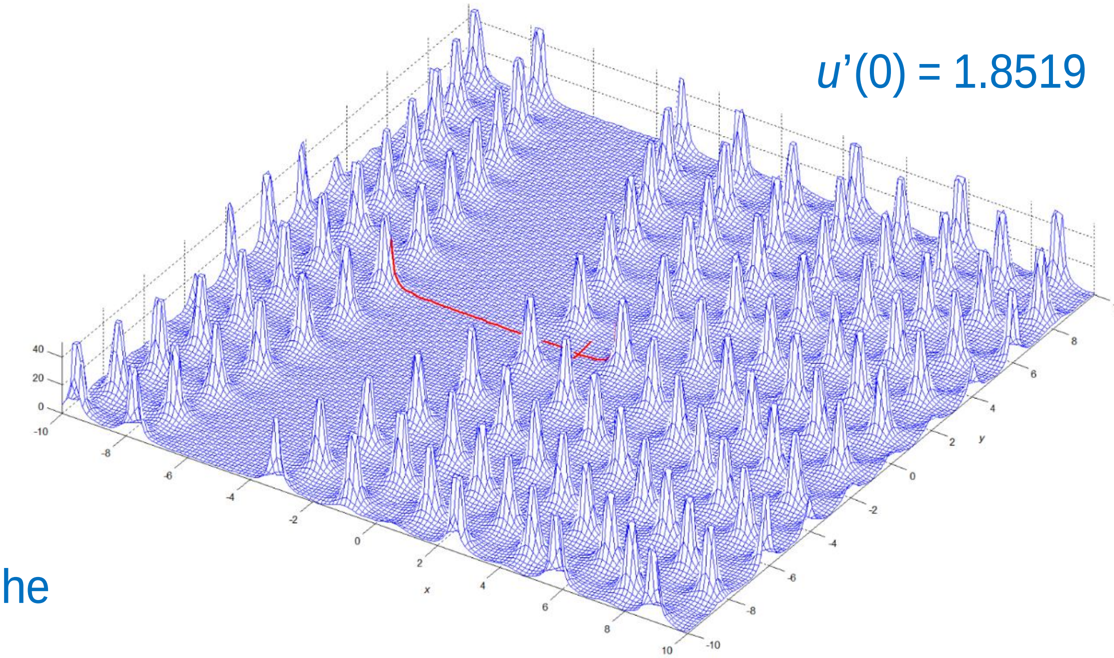
Illustration above:

Transition through a tronquée case as seen along the real axis using a standard ODE solver

$u'(0) = 1.8518$



$u'(0) = 1.8519$



Illustrations to the right:

The same transition computed by the *pole field solver .*

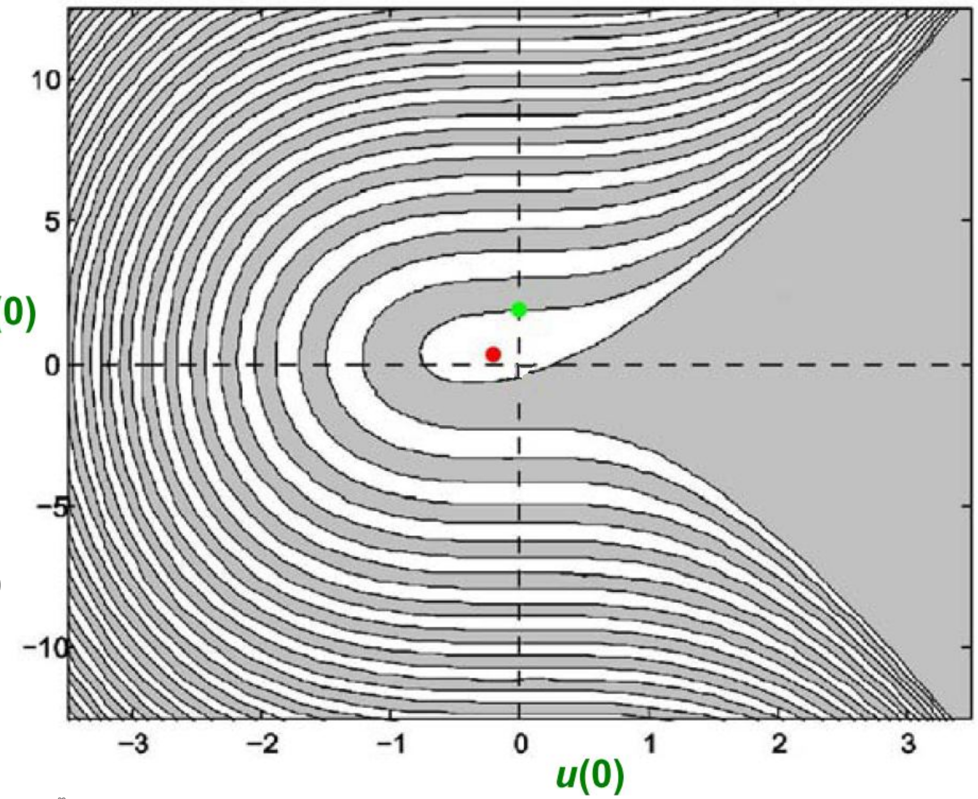
P_I: Initial conditions for u(0), u'(0) that give rise to tronquée solutions

Cases illustrated above:

Tritronquée: $u(0) \approx -0.188$, $u'(0) \approx 0.305$
NIST example: $u(0) \approx 0$, $u'(0) \approx 1.852$

White regions: Oscillations when $x \to -\infty$
Shaded regions: Poles when $x \to -\infty$

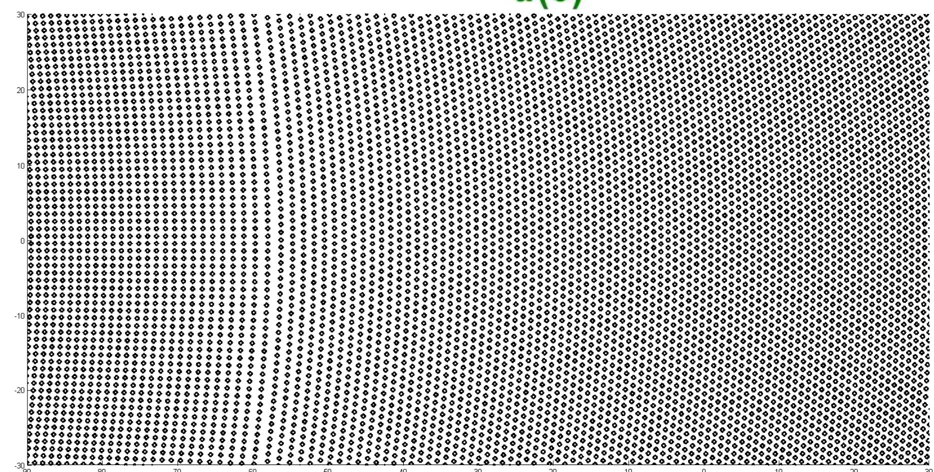
Black curves: Tronquée cases
(pole free sector in left half-plane)



Another pole field illustration:

Example of 'fracture line' within a pole field

IC: $u(0) = -5$, $u'(0) = 0$ (in white region above);
Pole field displayed over [-90,30]x[-30,30]



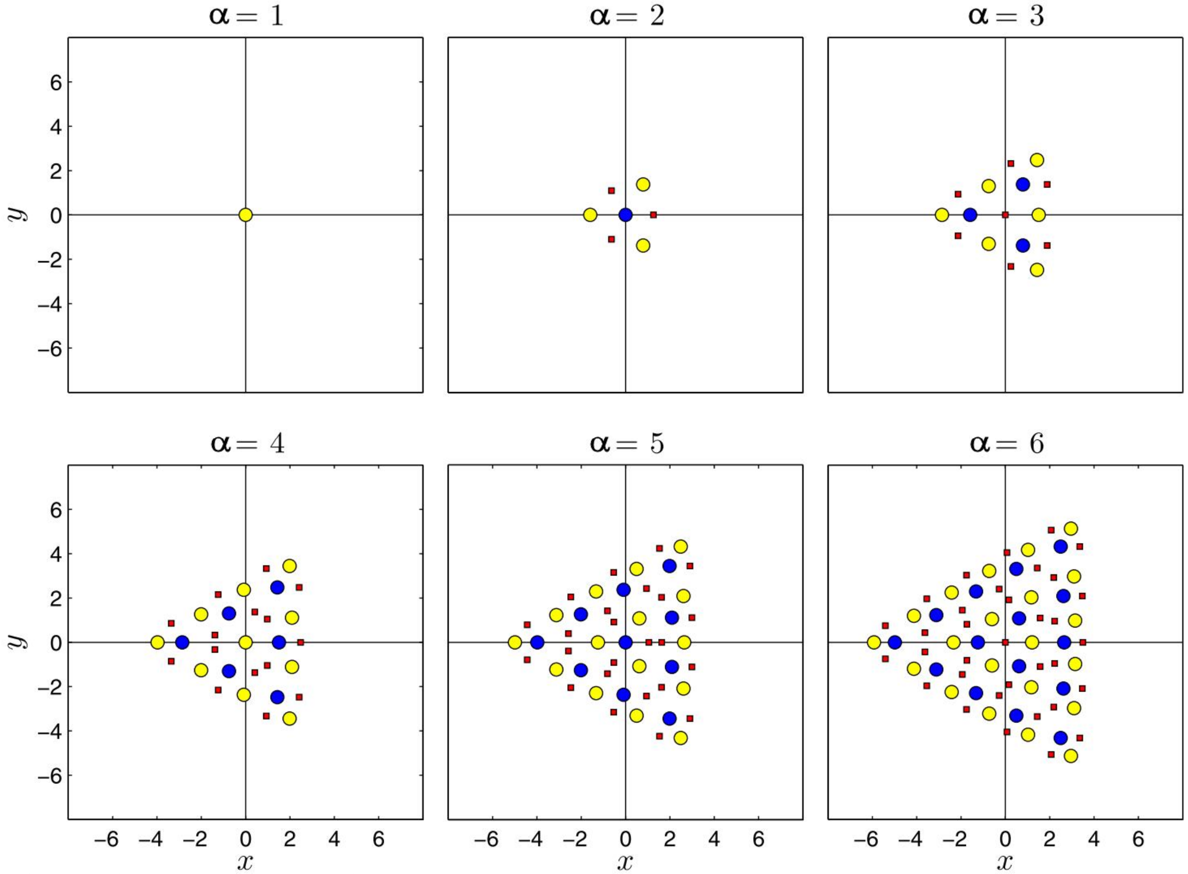
Brief survey of the solution space to the P_{II} equation

$$P_{II}: \quad \frac{d^2u}{dz^2} = 2u^3 + zu + \alpha$$

Three-parameter solution space: $(\alpha, u(0), u'(0))$; suffices to consider $\alpha \geq 0$.
Two types of closed form solutions known; represent discrete points or curves in the 3-D space

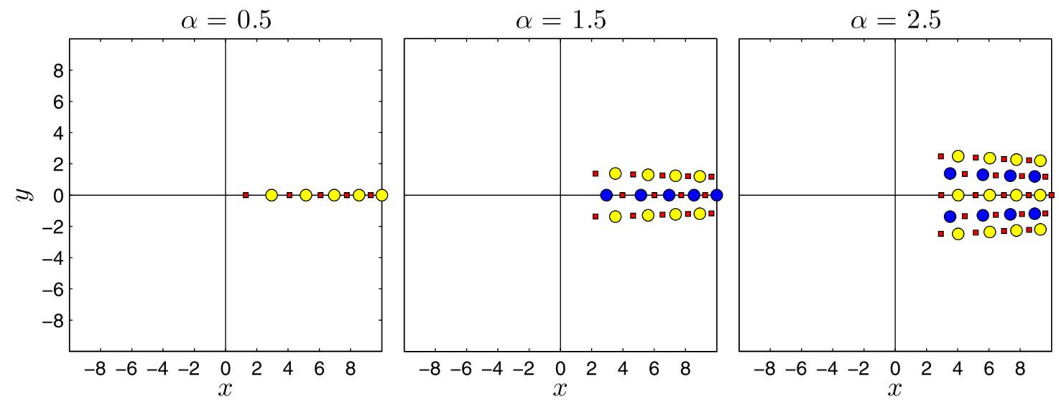
(i) Rational solutions when α integer:

Pole, residue +1 Blue
Pole, residue - 1 Yellow
Zero Red



(ii) Airy-type closed form solutions

Exist only for α 'half-integer'
Examples for α = 1/2, 3/2, 5/2 $\Rightarrow$

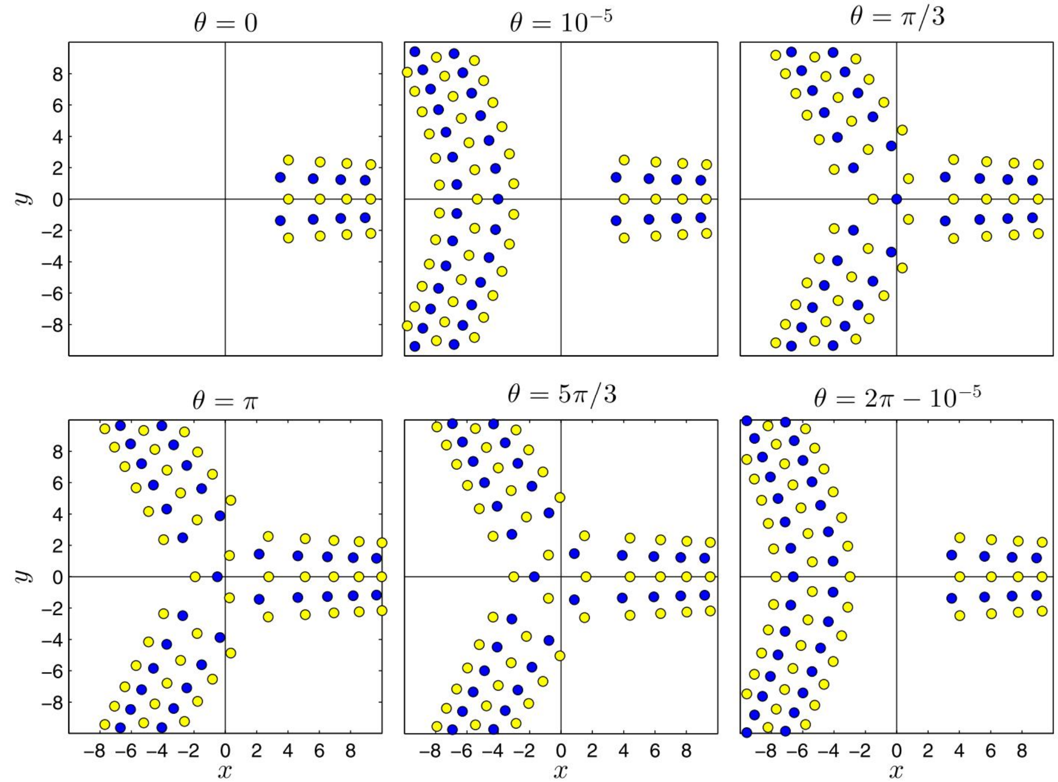


Each α-case extends to one parameter family of closed form solutions – shown here in the case of α = 5/2 $\Rightarrow$
(zeros not displayed here)

Three symmetry directions for $\theta = \pi/3$ and for $\theta = 5\pi/3$

The process picks up a 5-group of poles from the right region and brings it out to minus infinity

Closed form solutions provide merely 'glimpses' (non-typical cases) of the full 3-D solution space (e.g. in no case a pole field throughout a full sector)

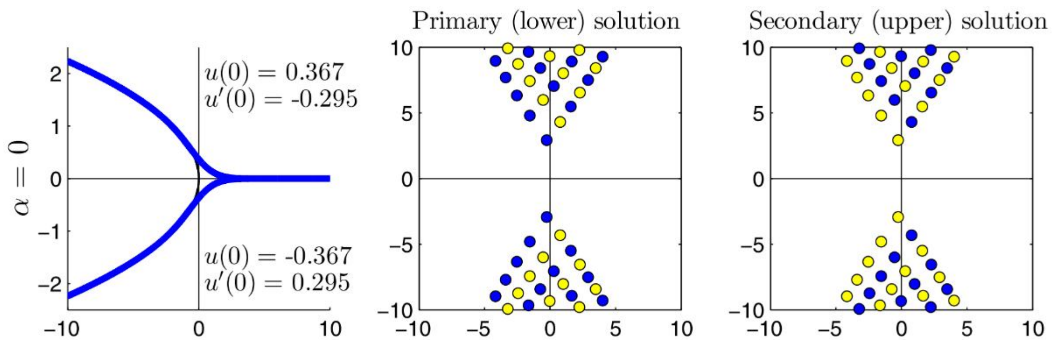


Solution space in the case of $\alpha = 0$

White: Infinity of poles on both R^+ and R^-
Grey: Infinity of poles on R^+ , oscillatory solution on R^-

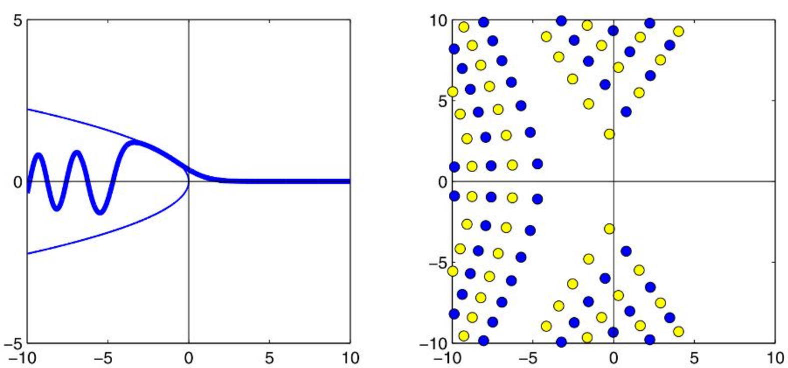
$\#^+$ Number of poles on positive real axis R^+ (curves)
$\#^-$ Number of poles on negative real axis R^- (regions)

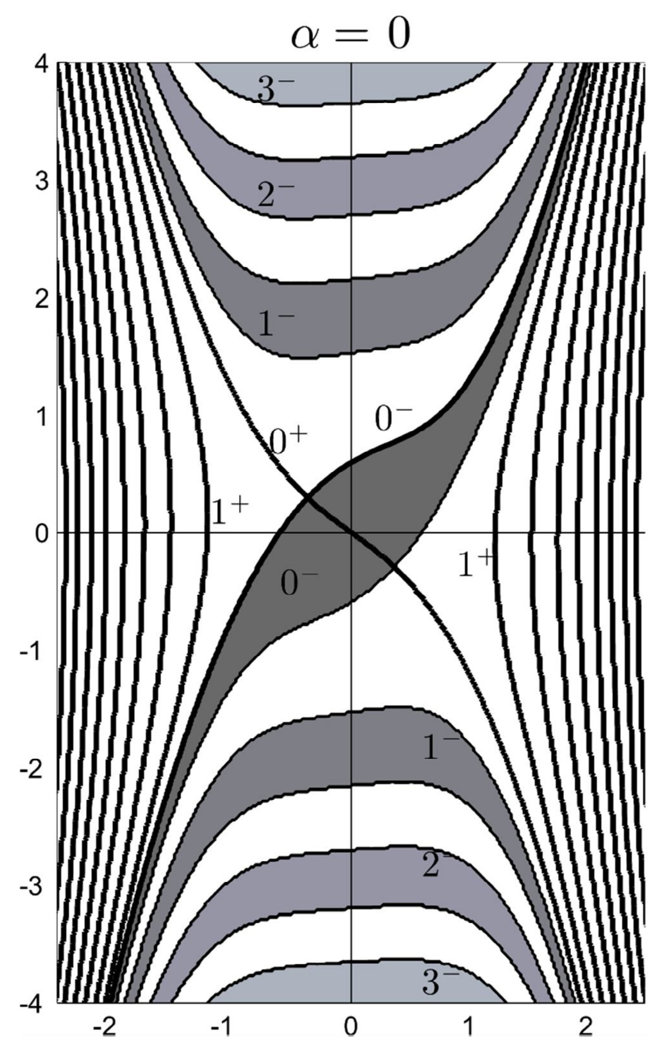
Hastings-McLeod: Intersection of 0^+ with edges of 0^-



Ablowitz-Segur: Intersection of 0^+ with interior of 0^-

Example near
'upper' solution:





Solution spaces for different α values

A wide range of solution 'dynamics' occur when α is increased. In particular:

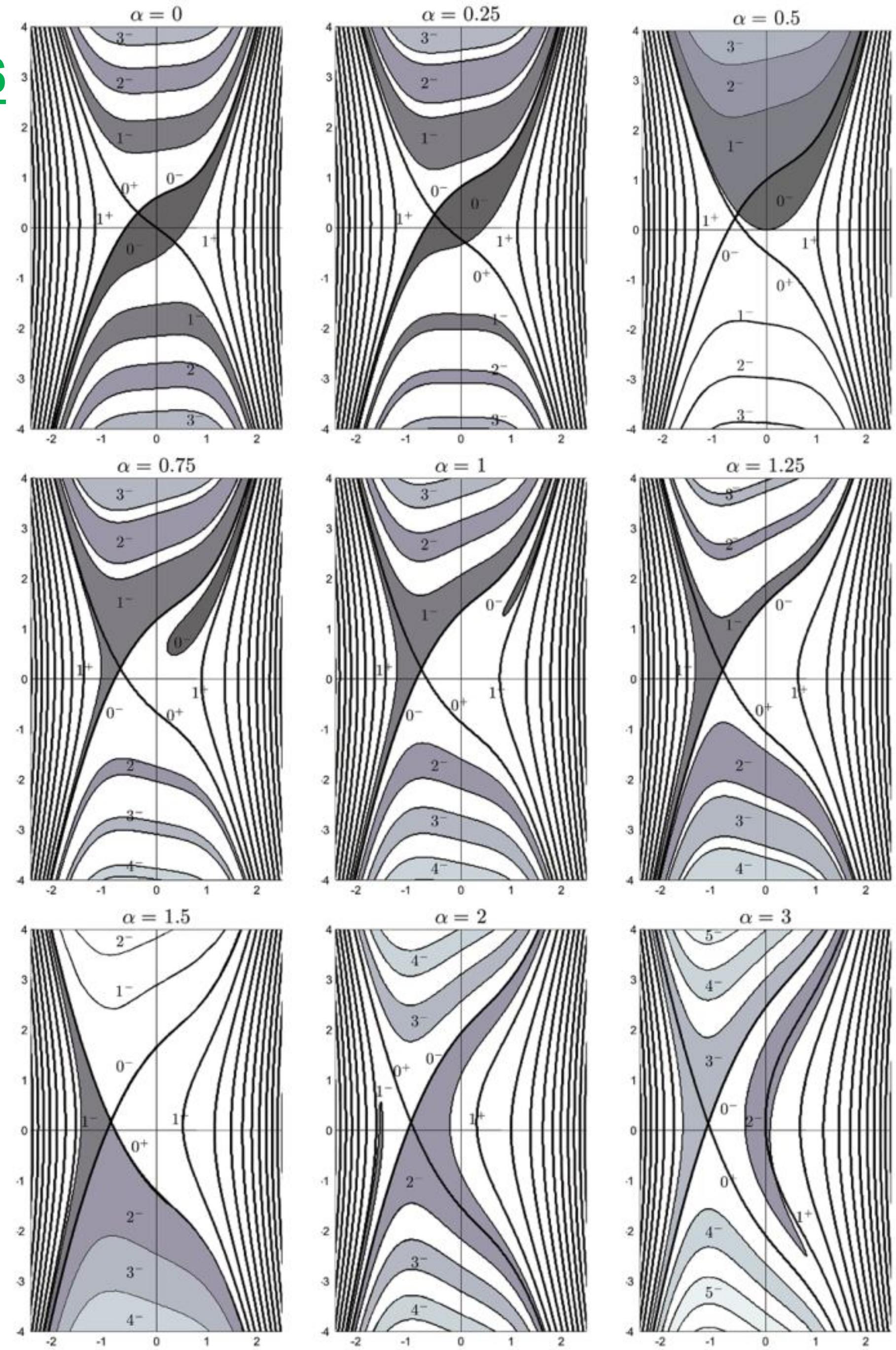
Only one H-ML and no AS solution for $\alpha \geq ½$

Beyond this, there appear 'generalized' H-ML and AS solutions, with finite number of poles on the real axis.

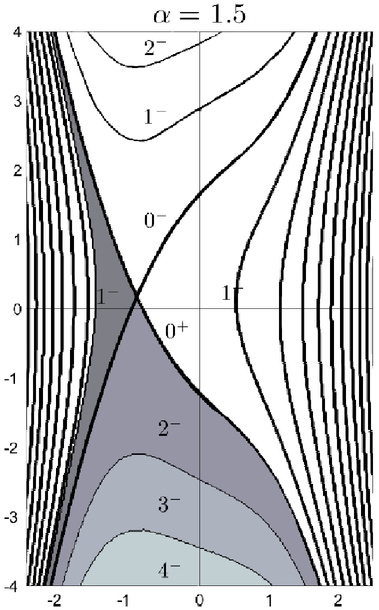
Edges of regions may have different character than regions on either side of it

Certain solution regimes vanish when α half-Integer

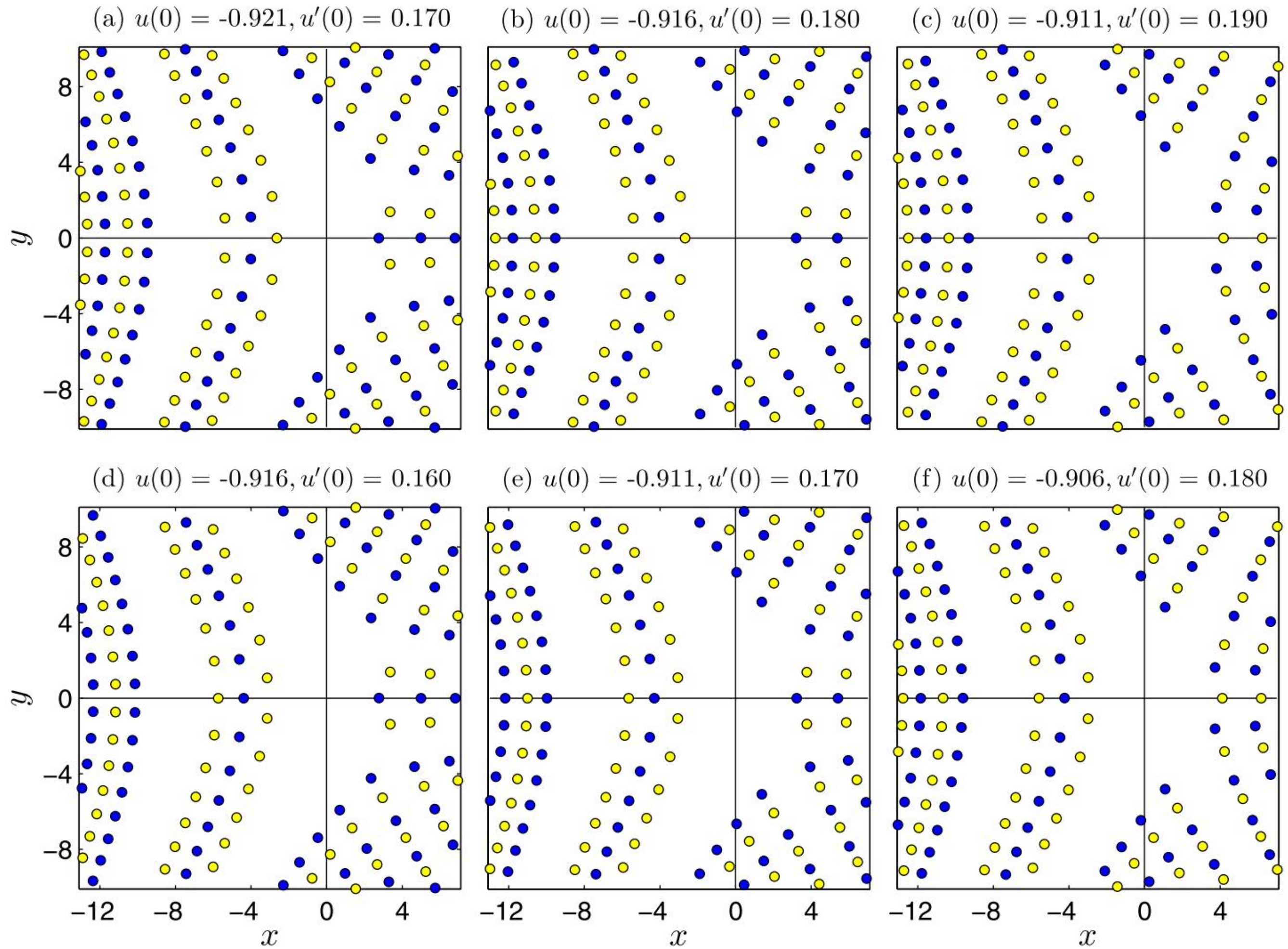
More details for the α = 1.5 case on next slide



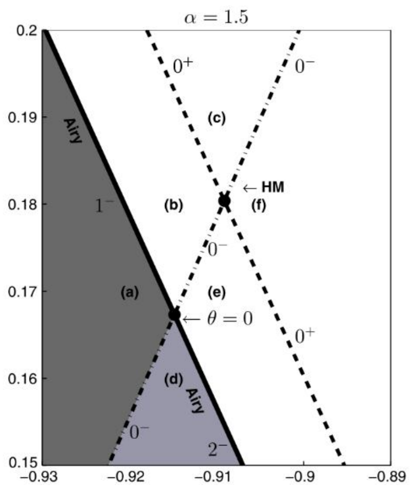
Another illustration of P_{II} 'solution dynamics'



Pole fields at the six locations marked (a) – (f):



Detail near H-ML point:



The imaginary P_{II} equation

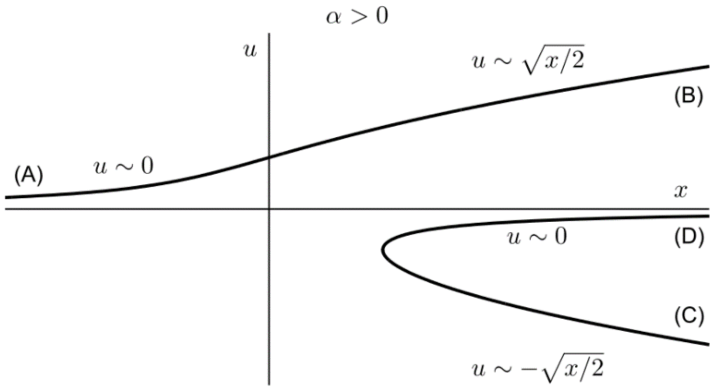
Regular P_{II}: $y'' = 2y^3 + zy + \beta$;

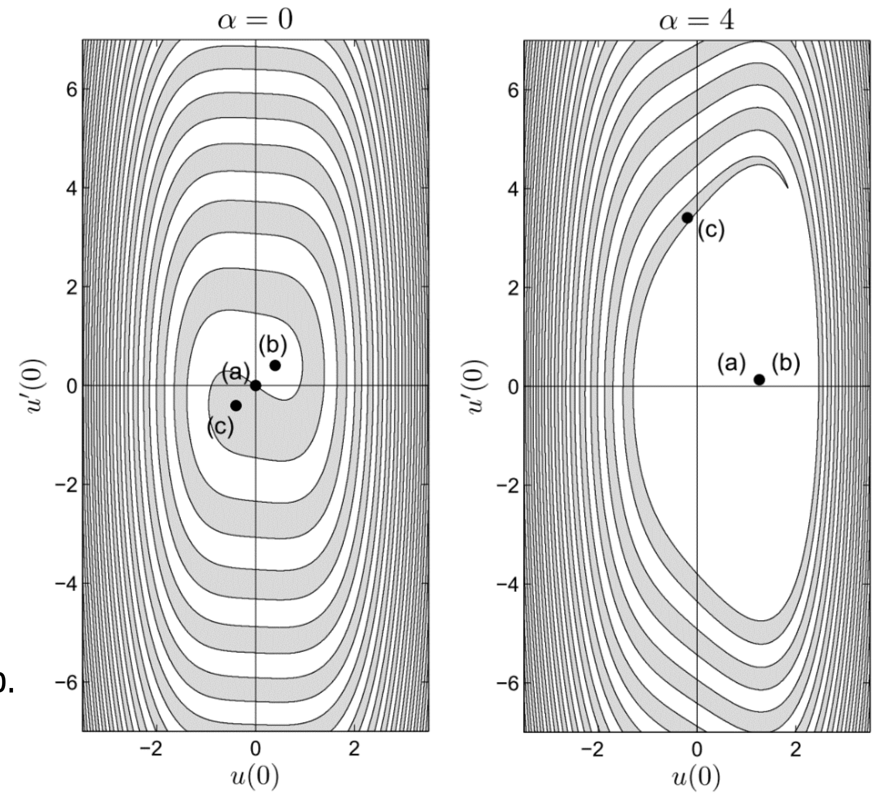
We have previously assumed $y(z)$ real for z real;
Modify this assumption to $y(z)$ imaginary for z real

Changing variables $y(z) = i\,u(z)$, $\beta = i\,\alpha$ gives the Imaginary P_{II} equation: $u'' = -2u^3 + zu + \alpha$

- No known closed form solutions (apart from the trivial $u(z) \equiv 0$ for $\alpha = 0$)
- All poles are first order and have residue $\pm i$ (as opposed to ± 1 for the regular P_{II})
- There are never any poles on the real axis.

Below: Possible asymptotes for solutions along the real axis



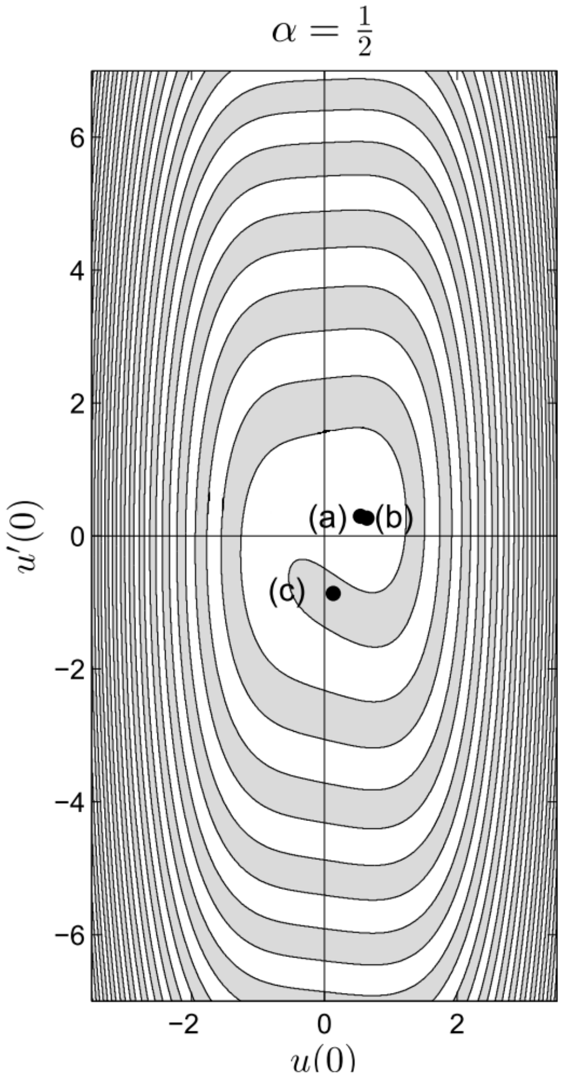


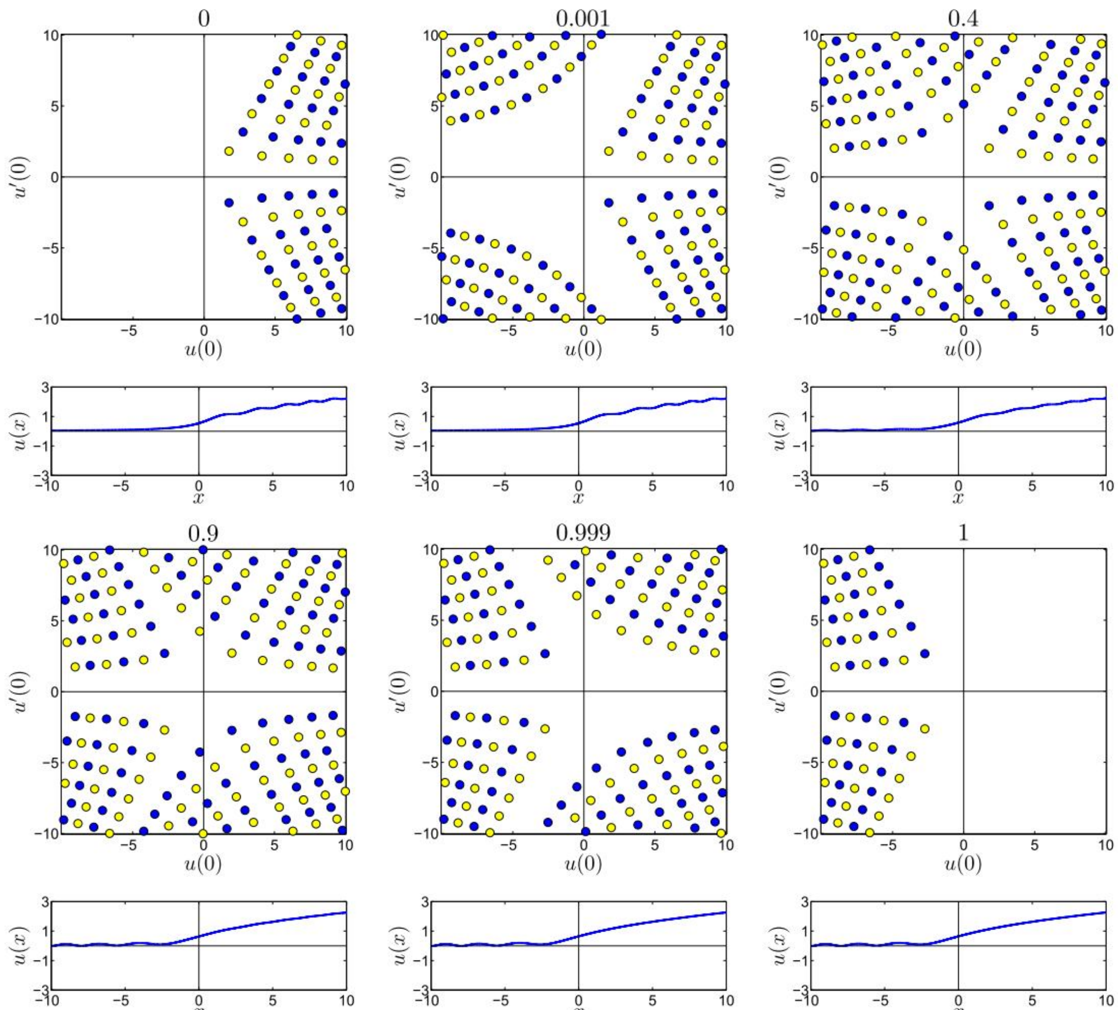
To the right: Asymptotic character for different ICs at $z = 0$.

(a,b,c): unique points: non-oscillatory conv. to (A,B,C) resp.
grey: Oscillatory convergence according to (C)
white: Oscillatory convergence according to (B)
black curves: Non-oscillatory convergence to (D)

Transition between the (a) and (b) cases when $\alpha = ½$.

A lot of ‘pole dynamics’ occurs which is not apparent from what is seen along the real axis



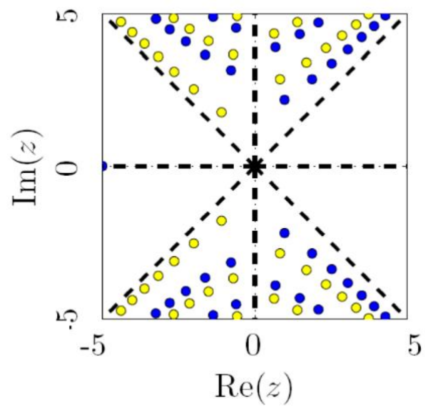


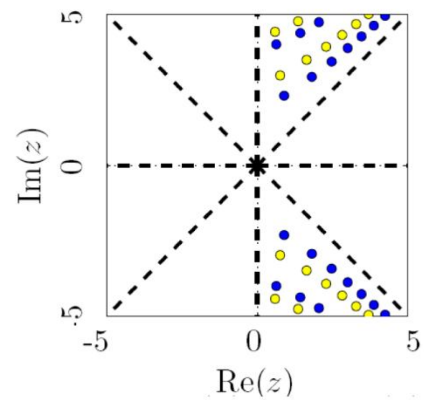
P_{IV}: The fourth Painlevé equation

$$P_{IV}: \quad \frac{d^2u}{dz^2} = \frac{1}{2u}\left(\frac{du}{dz}\right)^2 + \frac{3}{2}u^3 + 4zu^2 + 2(z^2-\alpha)u + \frac{\beta}{u}$$

- P_{IV} has two free parameters α and β (as well as the two ICs; four free parameters in all)
- Like for P_{II}, all poles are first order, with residues ± 1
- A variety of closed form solutions are known – but these are all ‘atypical’ cases in a much larger solution space. No closed form solutions are known for $\beta > 0$.

Already in the case of $\alpha = \beta = 0$ (right), computations reveal several families of tronquée-type solutions, including different cases that are smooth and non-oscillatory long the entire real axis.





For general α, β, there is a vast complexity of solution types / phenomena

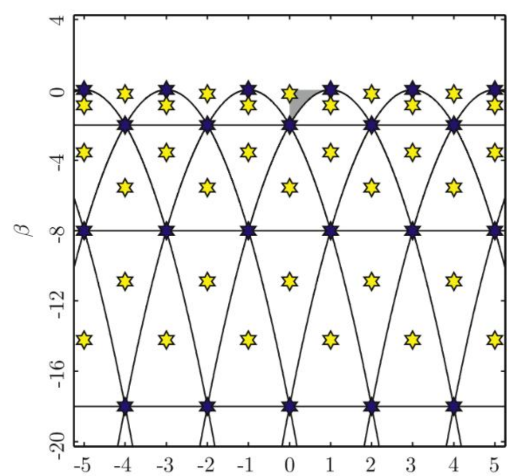
Curves and markers to the right indicate where closed form solutions exist in the α,β-plane for some choice of ICs

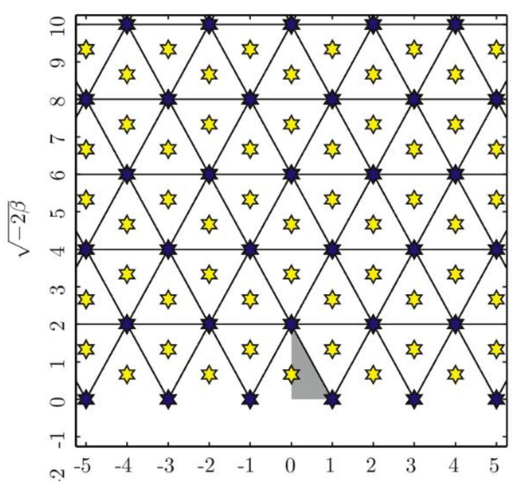
Grey: Weil Chambers'

✶ Generalized Hermite type

✶ Generalized Okamoto type

Curves: Parabolic cylinder and confluent hypergeometric types





Current project status

Completed work:

- Numerical ***pole field solver*** developed, and the solution space of P_I 'surveyed'.
  A numerical methodology for the Painlevé equations (B.F. and J.A.C. Weideman), J. Comp. Phys. 230 (2011), 5957-5973.

- Solution space of P_{II} 'surveyed'
  A computational exploration of the second Painlevé equation (B.F. and J.A.C. Weideman), Found. Comp. Math. 14 (2014), 985-1016.

- Solution space of the imaginary P_{II} equation 'surveyed'
  The solution space if the imaginary Painlevé II equation (B.F. and J.A.C. Weideman), submitted.

- Solution space of P_{IV} 'surveyed'
  Painlevé IV with both parameters zero: A numerical study (J.A. Reeger and B.F.), Stud. Appl. Math. 130 (2013), 108-133.
  Painlevé IV: A numerical study of the fundamental domain and beyond (J.A. Reeger and B.F.), Physica D. 280-281 (2014), 1-13.

- Numerical scheme tested successfully also on P_{III} , P_V and P_{VI} (J.A. Reeger, Ph.D. thesis, unpublished)

In Progress:

- Survey of the solution space to the P_{III} equation (M. Fasondini, J.A.C. Weideman and B.F.)