Some Observations Regarding Steady Laminar Flows Past Bluff Bodies

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Abstract

Steady laminar flows past simple objects, such as a cylinder or a sphere, have been studied for soon a century. During this time, theoretical, experimental and numerical methods have all contributed fundamentally towards our understanding of the resulting flows. The present article focuses on developments during the last few decades, when mostly numerical and asymptotical advances have provided insights also for steady although unstable high Reynolds numbers flow regimes.

1 Introduction

Viscous flows past blunt bodies become unstable at relatively low Reynolds numbers (Re). The first calculations that reliably continued steady solutions into such regimes were carried out about 30 years ago, revealing some

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unexpected trends. In the case of $Re = \infty$ (Euler flows), different computational and analytical tools provide a much larger number of solution options than those that correspond to plausible limits of $Re \to \infty$. In this article, we summarize computational results for these two cases, and quote some associated analysis. We conclude by noting some unresolved issues.

A detailed survey of the problems that we touch upon would require a much longer report than we are attempting here. However, by highlighting select results obtained by different methodologies, we hope to facilitate the exchange of ideas between specialists on different perspectives of the topic. Previous surveys specializing on numerical and asymptotic aspects of the present topic include [21] and [9], respectively.

2 Calculations of steady flows past a cylinder and a sphere at finite Re

2.1 Governing equations

Viscous, incompressible flows are described by the Navier-Stokes (NS) equations. In 2-D, these are most easily expressed in terms of stream function ψ and vorticity ω. With u and v denoting velocities in the x- and y-directions, respectively, it will hold that $u = \frac{\partial\psi}{\partial y}$ and $v = -\frac{\partial\psi}{\partial x}$. With vorticity defined by $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, the steady state NS equations become

$$\triangle\psi + \omega = 0, \tag{1}$$

$$\triangle\omega = \frac{Re}{2}\left\{\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}\right\}, \tag{2}$$

where $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The Reynolds number Re has here been defined according to the diameter of a cylinder with unit radius - hence the division by 2 in (2). In the special case of steady axisymmetric 3-D flows, (1) and (2) can be replaced by

$$\triangle\psi - \frac{1}{y}\frac{\partial\psi}{\partial y} + y\,\omega = 0, \tag{3}$$

$$\triangle\omega + \frac{1}{y}\frac{\partial\omega}{\partial y} - \frac{\omega}{y^2} = \frac{Re}{2y}\left\{\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} + \frac{\omega}{y}\frac{\partial\psi}{\partial x}\right\}. \tag{4}$$

Very low Re flows are known as *Stokes' flows* or creeping flows. For these, inertial terms become insignificant compared to viscous ones, and the flow field past a sphere approaches front-to-back symmetry when $Re \to 0$. For flow past a cylinder, *Stokes' paradox* tells that flows in this limit cannot match free stream at large distances, making the symmetry only approximate. These symmetries are rapidly lost when Re is increased. The flows first develop downstream recirculation regions (RRs) and shortly thereafter temporal instabilities, the latter for the cylinder and the sphere at $Re \approx 40$ and $Re \approx 105$, respectively. As noted above, the present article focuses on steady (although unstable) flows well past these Re numbers.

2.2 Computational issues

Several computational issues arise when computing steady high-Re flows. Some of these are discussed next.

2.2.1 Far field boundary conditions (BCs)

Focusing on the cylinder case, we will see in Section 2.3.1 that the vorticity will be confined to a thin downstream streak and decays exponentially fast when moving away from it. Imposing $\omega = 0$ just above the streak is entirely acceptable. The situation for ψ is different, as (1) closely couples the flow at the body surface (and, with it, the vorticity generation) to ψ-values far out in all angular directions. As Figures 1 a,b illustrate, BCs for either ψ or $\frac{\partial \psi}{\partial r}$ that do not somehow possess knowledge about the actual wake structure will have to be imposed extremely far out, suggesting computations throughout a large region in which the governing equations (1), (2) have simplified to $\triangle \psi = 0$, $\omega = 0$. The two uppermost curves in Figures 1 c illustrate more desirable outermost computational grid lines. As first utilized in [19], one can analytically create any number of 'sample' solutions that obey $\triangle(\psi - \psi_{FS}) = 0$ outside the computational domain, together with the appropriate BC at infinite distance, e.g. by placing point vortices in turn at the locations marked by circles (and their antisymmetric counterparts). Each case gives a linear relation that has to hold between the ψ values at the outermost and next-to-outermost grid lines. These relations, reminiscent of Robin-type BCs, remove the need for any numerical computations outside the domain sketched in part c of the figure. It will transpire that the downstream boundary is not critical, since errors in these will barely penetrate backwards against the flow.

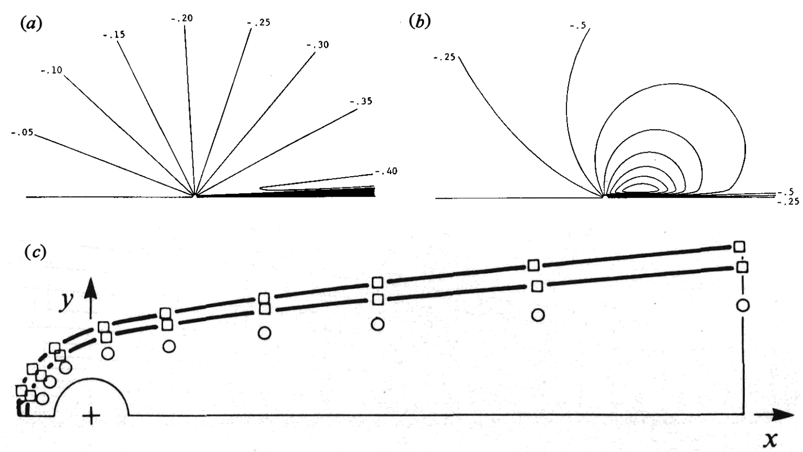

Figure 1: (a) Difference between free stream $\psi_{FS} = y$ and the flow within the region $1 \leq r \leq 60$ according to the leading term in the Oseen expansion [26, 27] at $Re = 200$ (assuming the correct value of the drag coefficient C_D at this Re), (b) Actual difference, (c) Concept illustration for obtaining a BC for ψ that is accurate already when used immediately outside the downwind vorticity streak. The different parts of the figure are copied from [18, 19], with permission.

2.2.2 Spatial discretization

Several discretization options are available. Once the computational domain has been limited to a region similar to the one shown in Figure 1 (c), finite elements (using unstructured meshes) and RBF-FD (which are mesh-free) [22, 37] can be applied directly. It has however been more common to conformally map the domain to a rectangle, and then apply nonlinear stretchings in the two directions to resolve the boundary layer, etc. Between the further choices of finite differences (FD) of different orders and pseudospectral approximations, the second order centered FD approximations used successfully in [17, 18, 19] were nevertheless quite certainly sub-optimal with regard to computational efficiency. Furthermore, given the vast improvements in computer power during the last 30 years, it is surprising that these early works have not yet been far surpassed.

2.2.3 Newton's method for obtaining steady solutions

Newton's method is well known both for scalar equations and for nonlinear systems, in both cases featuring quadratic convergence in the vicinity of simple roots. It was not fully appreciated until around 1980 that no adverse issues would arise when the system sizes were scaled up to the 10's or 100's of thousands of nonlinear equations that arise for challenging PDE discretizations. Compared to other fixed point iterations, the quadratic convergence provides some major advantages: (i) guaranteed low number of iterations, (ii) no possibility of time instabilities carrying over into the artificial time of successive iterations, and (iii) no problem with the fact that, on a solid boundary, there are two conditions for ψ and none for ω, since the total count of equations and unknowns remains correct. Homotopy (continuation) methods are readily available to effectively step in a parameter (such as Re). Also, with Newton's method, numerical procedures to handle bifurcation points are well established. As it happened, no such were encountered for either flow past a single cylinder or axisymmetric flow past a sphere, but cases where such arise are noted in Section 5.3.

2.3 Some steady state results

2.3.1 Flow past a cylinder

Noteworthy trends seen when Re is increased up to around $Re = 200$ include (i) the length L of the recirculation region (RR) increases, and (ii) although

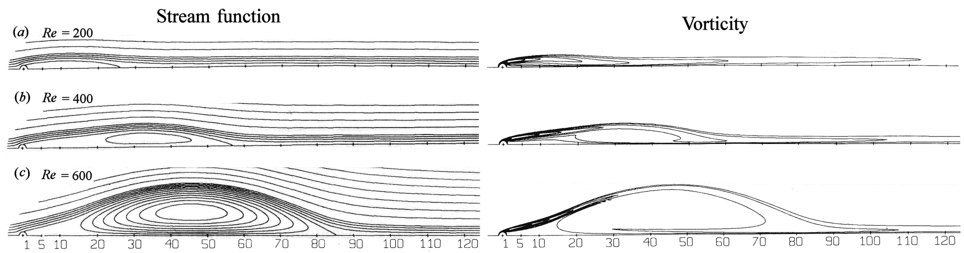

Figure 2: Stream function and vorticity in steady flows past a cylinder. The figure is adapted from [18], with permission.

the vortex streak gets thinner, dissipation is reduced and the streak 'survives' much further downstream. As seen in Figures 2 and 3, the latter trend overtakes the former around $Re = 400$, in the sense that vorticity then starts to be convected back into the RR from its back, and soon fills up its interior with nearly constant vorticity. Surprisingly, this transition has hardly any effect on the linear growth of L, but the width W of the RR then also starts to grow linearly with Re; cf. Figure 4. The sequence of dots labeled "$0.17Re$" corresponds to (slender) wake analysis in [40], whereas the significance of the dots labeled "$Length/1.6691$" will become apparent after the discussion of the Sadovski vortices in Section 3. We note that the slope of the dot sequence closely matches that for W.

In the case of a cascade of infinitely many side-by-side cylinders, an interesting further effect occurs in that there is a critical cylinder separation (with cylinder centres around 20 radii apart), below which this transition never occurs, no matter how high the Re is chosen. Above it, significant amounts of vorticity enters the RR from its rear, causing it to 'bulge' (although with W constrained by the cylinder separation). Cascade flows were considered in [20, 23, 31, 33, 41]. The fact that cascade configurations simplify far field boundary conditions is advantageous for analysis, but not essential for computation (cf. Section 2.2.1).

2.3.2 Flow past a sphere

Figure 5 illustrates the evolution of the RR for increasing Re, as computed in [19]. Differences from the cylinder case include (i) starting shortly before $Re = 1000$, a secondary RR emerges inside the primary one, (ii) in agreement with the Prandtl-Batchelor theorem, the vorticity inside the RR becomes

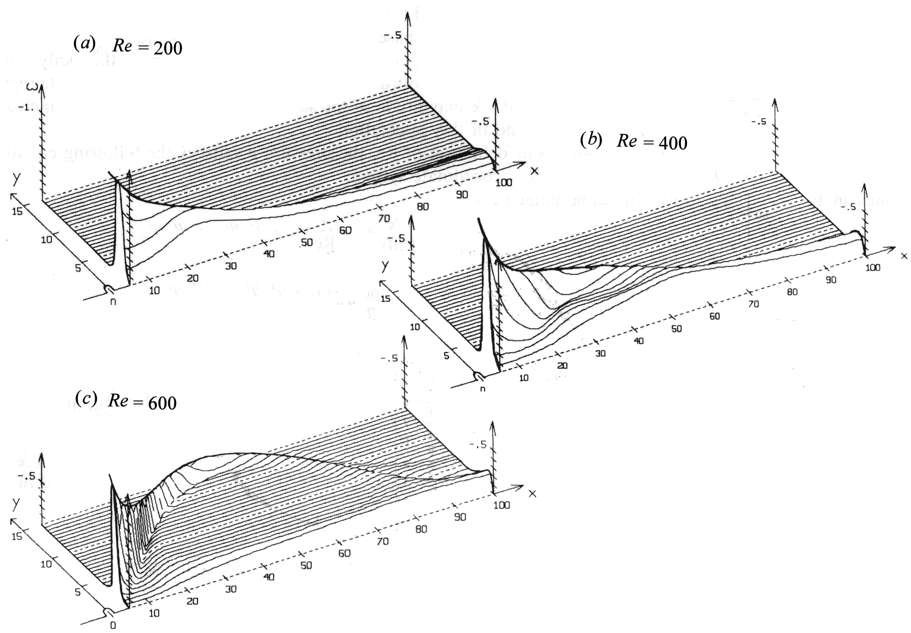

Figure 3: Surface display of the vorticity fields shown in Figure 2. The figure is adapted from [18], with permission.

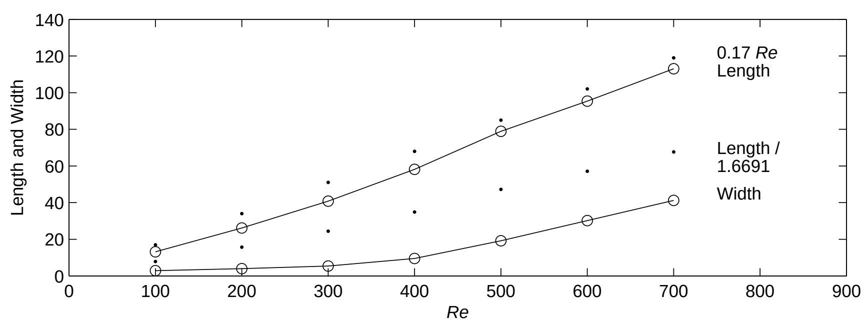

Figure 4: Growth of length L (measured from the cylinder centre) and width W of steady RR behind a cylinder, based on data from [20].

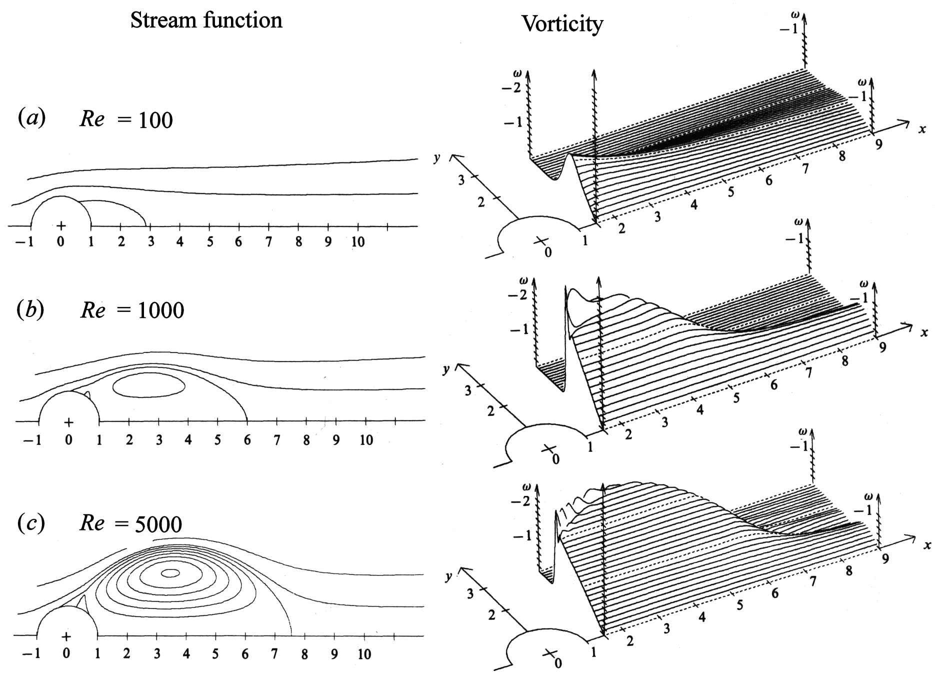

Figure 5: Stream function and downstream wake vorticity for steady flow past a sphere, at Re = 100, 1000, 5000. The figure is adapted from [19], with permission.

proportional to y (rather than to a constant), and (iii) the shape of the RR approaches the form of a Hill's spherical vortex (cf. Section 4).

3 Euler solutions for flows past a cylinder and a sphere

In seeking to understand high Re flows it is natural to consider Euler flows, formally corresponding to $Re = \infty$. Only those in which the vorticity somewhere is non-zero are of interest, since otherwise the action of viscosity in the history of the flow can not be felt, and deductions of non-physical results such as d'Alembert's paradox can be made. A discussion of relevant ideas predating many of the results surveyed here is given in [35] (written for the 25^{th} anniversary of the Journal of Fluid Mechanics). In the case of a circular cylinder in 2-D, the present authors believe that [14] identifies all single vortex flow options. These flows are naturally cataloged in terms of equilibrium point vortex flows as indicated in the Figures 6, 7. There

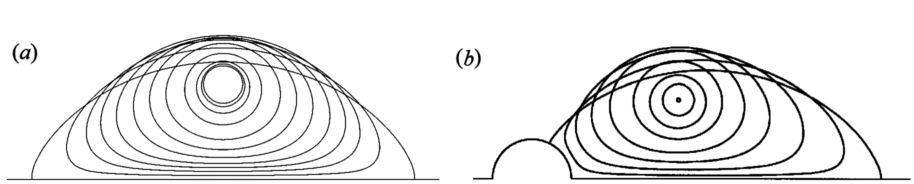

Figure 6: Sadovski type vortices (a) without any body and (b) behind a cylinder. These different vortices feature constant levels of vorticity inside the shown contours, and zero vorticity outside them.

is a natural continuation from point vortices with a given circulation to attached, Batchelor type vortices [1] (of finite size for increasing Re), and to families of vortices in which the support extends to infinity. All of the shown solutions were obtained by solving nonlinear free boundary Poisson equations for the stream function on a finite difference grid.

Equilibrium positions for point vortices played a central role in the works just quoted and also in studies by Zannetti and co-workers [11, 47]. The point vortices seem to indicate where less singular solutions also may exist. In general, steady Euler flows are too large a class for describing possible $Re \to \infty$ limits, and some criterion must be used to single out a solution. For example the use of a Kutta condition at corners or cusps on the physical boundary can be used and, in other problems, separation of the boundary layer can play a role.

The catalog of flows found in [14] extend to axisymmetric flows past a sphere. Although there is then no analog of the point vortex, some possible flows are shown in Figure 8. Flows including swirl around the axis, as described by the Bragg-Hawthorne equation, can also be found, cf. Figure 9. More illustrations and further discussion can be found in [16].

Of particular interest for large Re limits of the Navier-Stokes equations are the Sadovski vortex in 2-D (the curve reaching the center line in Figure 6 (a)) and Hill's spherical vortex in 3-D. The Sadovski vortex does not have a closed form expression, but it has been computed to high accuracy [36, 46]. It features the aspect ratio $L/W \approx 1.6691$. This is the case that is relevant for steady high Re flows past a cylinder, cf. Section 2.3.1 and Figure 4. In a generalized form, the Sadovski vortex can also have a jump in Bernoulli constant across its boundary. This vortex family has been computed in [32].

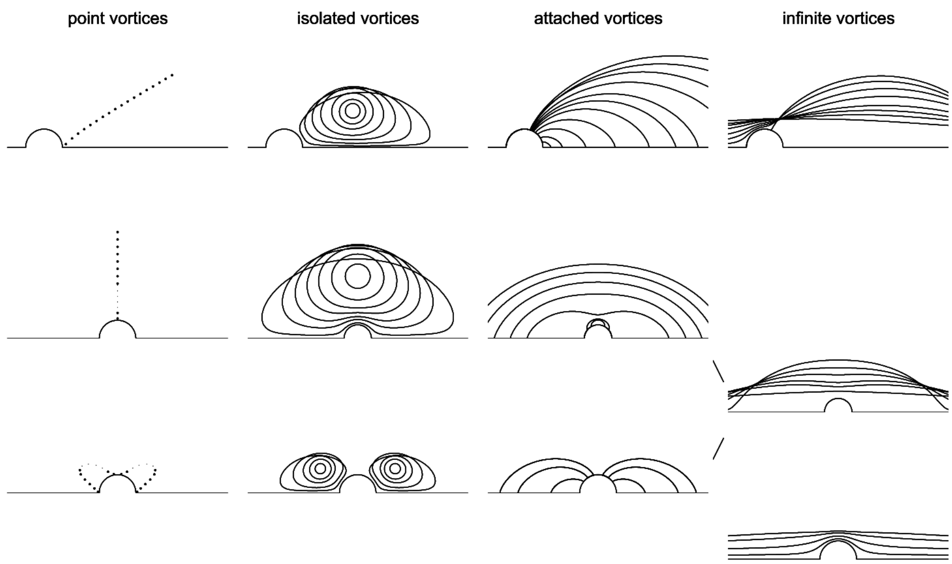

Figure 7: Illustrations of possible steady 2-D Euler flows. The figure is copied from [14], with permission.

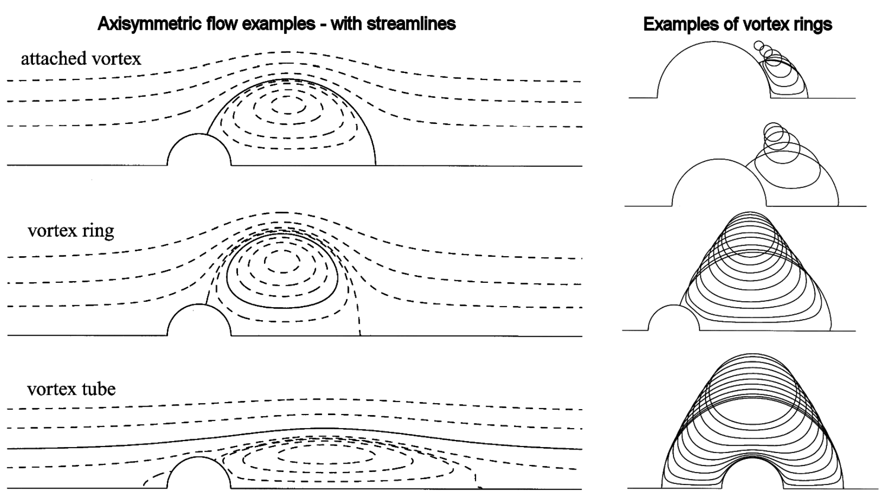

Figure 8: Some different types of axisymmetric flows. The figure is adapted from [15], with permission.

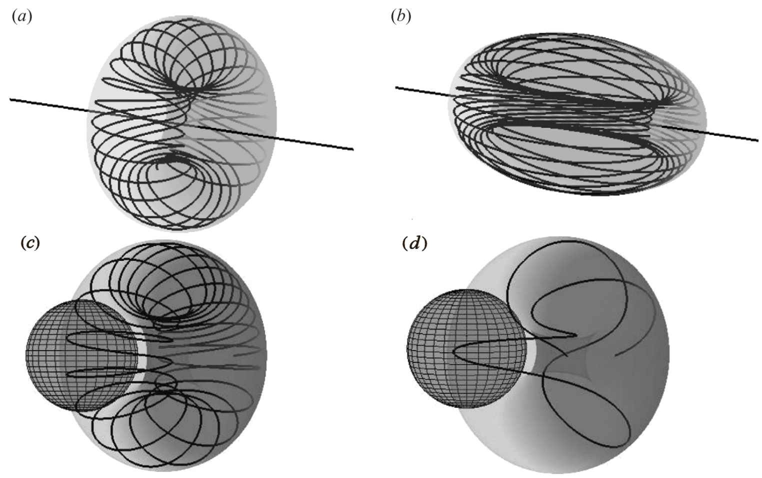

Figure 9: (a), (b) Vortex rings with swirl without and with shear in the exterior flow; (c), (d) Two vortex rings with swirl in flow past a sphere. The figure is adapted from [16], with permission.

4 Some asymptotic analysis results

The first building block in creating Euler solutions which might play a useful role in realistic flow models was given by Helmholtz [45], introducing vortex sheets separating from the body. Brodetsky [4] showed how this could be done also in the case of flow past a circular cylinder. A geometric condition determined the separation point, and a wake opening up parabolically to infinity was found. A series of theoretical studies starting with Squire [42] supported the idea of a “slender” elliptic wake tending for increasing Re towards Brodetsky’s free streamline model.

A detailed asymptotic theory of separation for the Navier-Stokes equations was given by Smith [39], further developing results in [43]. These works effectively incorporated the “triple deck” model for local separation; see [3] for its history. Batchelor [1] had earlier introduced the idea of a cyclic boundary layer around a recirculating eddy, which was analyzed further in [6, 29]. Various attempts were made to model the scale of the RR with Re, often predicting the length L to increase linearly with Re and the width W to increase as $Re^{1/2}$, consistent with the infinite Re parabolic wake of Brodetsky.

The first to propose a theory in which both L and W increased linearly with

Re was Taganov [44]. The computations of Fornberg described above confirmed this concept by showing a change in RR character at *Re* around 400, after which this wide wake is observed. The first complete self consistent theory was subsequently given by Chernyshenko [7], in excellent agreement with these numerical results. In this model the asymptotic separation analysis was matched to a boundary layer surrounding a steady Sadovski vortex, which grows linearly with Re. There are several distinguished limits in the matching process, and it is remarkable that a complete determination of the relevant parameters is possible. In the case of cascades of bodies, the corresponding asymptotic theory [10] is again in good agreement with the numerical results cited in Section 2.3.1. At least as far as flows past symmetric bodies in 2-D is concerned, it can be said [9] that "... the high Reynolds number asymptotics of steady plane flow past a bluff body is known."

For the analogous axisymmetric flow past a sphere, the calculation in [19], described above in Section 2.3.2, revealed a large wake in the shape of a scaled Hill's spherical vortex. Some asymptotic results similar to the cylinder case can again be given [8], but the secondary separation complicates the analysis. The problem of high *Re* asymptotics is still partly open, including how the RR radius scales with *Re*.

5 Open questions and challenges

The following is just a small collection of issues that seem especially intriguing, given the discussion and observations above.

5.1 Navier-Stokes solutions

Given that flows past asymmetric obstacles necessarily become asymmetric, it becomes natural to ask about the properties also of such solutions for large *Re*. If the obstacle shrinks to a point, it might seem that its geometric details should not have significant effect on the scale of the wake. On the other hand, present asymptotic theory makes essential use of symmetry assumptions. It has been conjectured (Chernyshenko, personal communication) that, for a pair of counter rotating vortices attached to an obstacle, one of these may persevere and the other shrink as *Re* grows. Suggestive results supporting this conjecture are obtained in [12] for an ellipse at 45^o angle of attack. However, the largest *Re* used there is $Re = 40$, and this is too low to draw definite conclusions. It would be interesting to obtain results also for significantly larger *Re*.

5.2 Euler flows

The authors are not aware of any asymmetric Prandtl-Batchelor flows with pairs of counter rotating vortices attached to the body. It seems likely that they may exist, at least if the geometry is a mild perturbation of symmetry, but none seem to have been described. There are solutions with a single vortex, for example the 'trapped vortices' used in the project VortexCell2050 (as documented at several web sites most easily found by searching on the project name).

5.3 Existence and uniqueness

The flows described above had symmetry about the centre line built into the solution process. It is natural to ask if there can be more than one symmetric solution at a given Re. This is especially interesting in view of the wealth of possible Euler solutions and various conjectures about large Re limits [35]. The solution branches described in [18, 19] were monitored for bifurcation points, but none were found. This does of course not rule out altogether separate solution branches, or bifurcation points outside the considered parameter regimes. Instances of non-unique symmetric cascade flow solutions were reported in [5]. However, the author of that work also noted that their physical reality was unclear, since they could only be seen on coarse grid and not on fine grid calculations. When the assumption of symmetry in the flow is dropped, asymmetric steady solutions have been found both for flow past a single sphere [28, 37] and for flow past a pair of cylinders [2, 48]. Many issues with regard to solution existence and uniqueness deserve further study.

5.4 Time instabilities and flow control

As noted above, steady symmetric flow past both a cylinder and a sphere appear to be unique for all values of Re. Without the constraints of 'steady' and 'symmetric', a variety of instability modes enter when Re is increased. In the cylinder case, the first is a Hopf bifurcation, leading to vertical oscillations towards the end of the RR, producing a Kármán vortex street in the wake. Because the end of a steady RR is a stagnation point, a small vertical plate at this location will have very little effect on the flow, apart from being highly effective in suppressing this particular mode [24]. However, such 'passive' flow control approaches will be unable to control subsequent instability modes (with the initial laminar ones analyzed in [30]). Flow past a 2-D cylinder and over backwards steps have served as test problems for several active flow control strategies, both for the finite Re case [13, 38] and for Euler (point vortex) flows [34].

Extending to 3-D, Figure 10 compares the drag for steady vs. unsteady flows in the case of a sphere. This illustrates one reason why it can be very desirable to operate in steady although unstable flow regimes - a blunt body can then move through a fluid as easily as a streamlined one. The 'drag crisis' (sudden drop in drag by an order of magnitude) seen around $Re = 4 \cdot 10^5$ corresponds to the unsteady wake changing character from being dominated by large laminar vortices to taking the form of a narrower turbulent streak. An example of passive flow control for this case is provided by the dimples on a golf ball, shifting this transition to slightly lower Re, thereby making the lower drag regime available in actual play.

Active flow control has been developed successfully for several 'real life' applications, most notably allowing military aircraft to fly in unstable regimes, offering lowered drag as well as increased maneuverability. However, both cost and complexity of such systems can be high.

Beyond sucking away boundary layers on the surface of an airfoil (potentially beneficial for delaying or eliminating separation as well as reducing some turbulence effects), attempting to control turbulent instabilities by direct feedback mechanisms encounters fundamental limitations already in 2-D [25]. Nevertheless, the possibility that steady flows can feature much better characteristics than unsteady ones provides a major motivation for their investigation.

6 Conclusions

We have reviewed some computational results and algorithms for the steady Navier-Stokes equations and related Euler flows. For flow past a cylinder the RR starts around $Re = 400$ to grow also in width proportionally to Re. Both numerical solutions and asymptotic analysis indicate that the RR thereafter can be described by a scaled Sadovski vortex. There are similar results for flow past a sphere with the Sadovski vortex replaced by Hill's spherical vortex, but here there is a secondary separation and asymptotic results are not complete.

There are rich families of Euler flows past a cylinder and a sphere. In some cases physically significant solutions can be singled out with extra

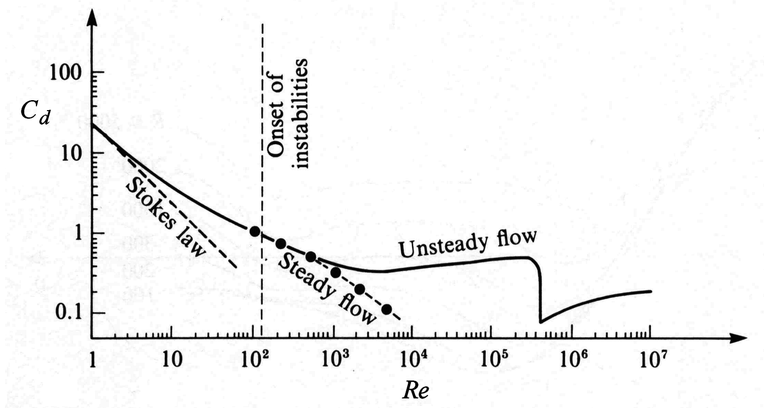

Figure 10: Drag coefficient C_d vs. Re for flow past a sphere. The figure is copied from [19], with permission.

conditions, such as Kutta conditions at corners. Some Euler flows can only occur as quasi steady limits of solutions of the Navier-Stokes equations, approximating flows for some period of time.

When the body is symmetric there is no firm evidence of non uniqueness for symmetric solutions, but certain asymmetric solutions have been found. When the body is not symmetric, as, for example, for flow past an ellipse at angle of attack so that the flow can not be symmetric, little is known about solutions for large Re.

Although steady flows become unsteady as Re grows, there is substantial motivation for finding ways to stabilize them, primarily in order to reduce drag.

Acknowledgments:

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