

Painlevé IV: A Numerical Study of the Fundamental Domain and Beyond

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The six Painlevé equations were introduced over a century ago, motivated by rather theoretical considerations. Over the last several decades, these equations and their solutions, known as the Painlevé transcendents, have been found to play an increasingly central role in numerous areas of mathematical physics. Due to extensive dense pole fields in the complex plane, their numerical evaluation remained challenging until the recent introduction of a fast ‘pole field solver’ (Fornberg and Weideman, J. Comp. Phys. 230 (2011), 5957-5973). The fourth Painlevé equation has two free parameters in its coefficients, as well as two free initial conditions. After summarizing key analytical results for P_{IV} , the present study applies this new computational tool to the the fundamental domain and a surrounding region of the parameter space. We confirm existing analytic and asymptotic knowledge about the equation, and also explore solution regimes which have not been described in the previous literature.

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I. INTRODUCTION

The solutions of the six Painlevé equations (P_I - P_{VI}) are free from movable branch points, but with the possibility of movable poles or movable isolated essential singularities (Ref. 1, Section 32.2). This Painlevé property inspired the introduction of a novel numerical approach²—combining a Padé based ODE solver³ with a partly randomized integration path strategy—and allowed for the first time rapid numerical solutions of the Painlevé equations over extended regions in the complex plane. It was first used for P_I ² and later for P_{II} ⁴. It was then applied to the fourth Painlevé equation

$$\frac{d^2}{dz^2}u(z) = \frac{1}{2u(z)} \left(\frac{d}{dz}u(z) \right)^2 + \frac{3}{2}u(z)^3 + 4zu(z)^2 + 2(z^2 - \alpha)u(z) + \frac{\beta}{u(z)}, \quad (1)$$

in the special case of $\alpha = \beta = 0$ ⁵. As in these three previous numerical studies, computational explorations in this paper are limited to solutions $u(z)$ that are real when z is real, although some of the presented theory includes solutions that are not always real on the real axis.

For a small portion of the two-dimensional (α, β) -parameter space there exist examples of solutions expressible as rational functions or in terms of special functions, such as the parabolic cylinder function. These well documented solutions appear, however, as only isolated points or one-parameter families in the four-dimensional space of parameters and initial conditions (ICs). Much of the present study is focused on the distribution of singularities for solutions to (1). These are all first order poles with residue +1 or -1.

A. Organization of the paper

Section 2 recalls some available theory, including symmetries in P_{IV} and different solution transformations. Section 3 discusses closed form solutions of P_{IV} , in particular solutions in terms of rational and elementary special functions and also the asymptotic behaviors presented in the literature. This is followed in section 4 by the numerical approach used here to explore the much larger space of solutions for which no closed form solutions are available. Sections 5 and 6 describe such explorations of the parameter and solution spaces, first highlighting the “fundamental domain” and then extending into inspections of the previously unexplored region of $\beta > 0$, for which no instances of closed form solutions or

transformations have been reported.

II. SYMMETRIES AND SOLUTION HIERARCHIES

This section describes the known symmetries in the P_{IV} equation and transformations that relate solutions for different parameter choices.

A. Symmetries in the Equation

Let $P_{IV}(\alpha, \beta)$ be the set of all solutions of (1) for the particular α and β . Direct inspection of (1) shows that if $u(z) \in P_{IV}(\alpha, \beta)$, then⁶

$$-u(-z) \in P_{IV}(\alpha, \beta), \quad (2)$$

$$-iu(-iz) \in P_{IV}(-\alpha, \beta), \text{ and} \quad (3)$$

$$iu(iz) \in P_{IV}(-\alpha, \beta). \quad (4)$$

Incidentally the first of these symmetries also holds for P_{III} (for all parameter choices), but never for any of the other Painlevé equations. It is important to keep these symmetries in mind since any solution presented in this paper has at least one other counterpart for the same choice of α and β .

B. The Bäcklund and Schlesinger Transformations

It is known that P_{II} through P_{VI} have collections of transformations relating solutions for given parameters to those of different choices. For instance, Refs. 6, 7, 8, 9, and others relate solutions $u(z) \in P_{IV}(\alpha, \beta)$ to $u_{k,\mu}^\pm(z) \in P_{IV}(\alpha_{k,\mu}^\pm, \beta_{k,\mu}^\pm)$, $k = 1, 2, 3$ through the relationships (5) through (7). Confining this study to solutions that are real on the real axis limits these

transformations to nonpositive β . These transformations are

$$u_{1,\mu}^{\pm}(u(z), z) = \frac{1}{2\mu u(z)} \left(\frac{d}{dz} u(z) \mp \mu(u(z)^2 + 2zu(z)) - \mu\sqrt{-2\beta} \right) \quad (5)$$

$$u_{2,\mu}^{\pm}(u(z), z) = \frac{\left(\frac{d}{dz} u(z) \pm \mu\sqrt{-2\beta}\right)^2 + (4\alpha + 4\mu \mp 2\sqrt{-2\beta})u(z)^2}{2u(z) \left(u(z)^2 + 2zu(z) - \mu\frac{d}{dz} u(z) \mp \sqrt{-2\beta}\right)} - \frac{u(z)^2(u(z) + 2z)^2}{2u(z) \left(u(z)^2 + 2zu(z) - \mu\frac{d}{dz} u(z) \mp \sqrt{-2\beta}\right)} \quad (6)$$

$$u_{3,\mu}^{\pm}(u(z), z) = u(z) + \frac{2 \left(1 - \mu\alpha \mp \frac{1}{2}\mu\sqrt{-2\beta}\right) u(z)}{\frac{d}{dz} u(z) \pm \mu\sqrt{-2\beta} + \mu(2zu(z) + u(z)^2)} \quad (7)$$

where $\mu = \pm 1$.

The transformed solutions $u_{k,\mu}^{\pm}$, $k = 1, 2, 3$ occur for the parameter choices

$$\alpha_{1,\mu}^{\pm} = \frac{1}{4}(\pm 2\mu - 2\alpha \pm 3\sqrt{-2\beta}) \text{ and } \beta_{1,\mu}^{\pm} = -\frac{1}{2} \left(1 \pm \alpha\mu + \frac{1}{2}\mu\sqrt{-2\beta}\right)^2 \quad (8)$$

$$\alpha_{2,\mu}^{\pm} = \alpha + \mu \text{ and } \beta_{2,\mu}^{\pm} = -\frac{1}{2}(2 \mp \mu\sqrt{-2\beta})^2 \quad (9)$$

$$\alpha_{3,\mu}^{\pm} = \frac{3}{2}\mu - \frac{1}{2}\alpha \mp \frac{3}{4}\sqrt{-2\beta} \text{ and } \beta_{3,\mu}^{\pm} = -\frac{1}{2} \left(\mu - \alpha \pm \frac{1}{2}\sqrt{-2\beta}\right)^2. \quad (10)$$

There are also composite transformations $u_4^{\pm} = u_{2,\pm}^+(u_{2,\pm}^-(u(z), z), z)$ and $u_5^{\pm} = u_{2,\mp}^+(u_{2,\pm}^-(u(z), z), z)$ discussed in Refs. 7 and 9. As noted in Ref. 10, $u_{2,+1}^-$ was not always presented correctly in previous literature.

III. CLOSED FORM SOLUTIONS AND ASYMPTOTIC APPROXIMATIONS

Before discussing the closed form solutions and asymptotic behaviors presented in the literature, note again that, even for choices of α and β admitting these solutions, little is known aside from at an isolated location or along one-parameter family of points in the two-dimensional plane of ICs.

Generalized Hermite Polynomial Type Solutions, $u_{m,n}^{[GH;k]11}$				
k	α	β	Special Choice	Special Solution
1	$2m + n + 1$	$-2n^2$	$m = 0, n = 1$	$\frac{1}{z}$
2	$-(m + 2n + 1)$	$-2m^2$	$m = 1, n = 0$	$-\frac{1}{z}$
3	$n - m$	$-2(m + n + 1)^2$	$m = 0, n = 0$	$-2z$

TABLE I. Parameter choices leading to solutions of P_{IV} expressible in terms of Generalized Hermite polynomials.

Generalized Okamoto Polynomial Type Solutions, $u_{m,n}^{[OK;k]11}$				
k	α	β	Special Choice	Special Solution
1	$2m + n$	$-2(n - \frac{1}{3})^2$	$m = 0, n = 0$	$-\frac{2}{3}z$
2	$-m - 2n$	$-2(m - \frac{1}{3})^2$	$m = 0, n = 0$	$-\frac{2}{3}z$
3	$n - m$	$-2(m + n + \frac{1}{3})^2$	$m = 0, n = 0$	$-\frac{2}{3}z$

TABLE II. Parameter choices leading to solutions of P_{IV} expressible in terms of generalized Okamoto polynomials.

A. Rational Solutions

The fourth Painlevé equation has six different sequences of parameter choices leading to rational solutions expressible in terms of either Generalized Hermite or Generalized Okamoto polynomials (see, e.g., Ref. 11), with two particular choices leading to the nontrivial entire solutions $-2z$ and $-2/3z$. Tables I and II state the choices leading to such solutions when $m, n \in \mathbb{Z}^+$. These locations in the (α, β) -plane will later be shown in figure 1 as dark (blue) and light (yellow) hexagrams for Generalized Hermite and Generalized Okamoto polynomials, respectively.

B. Special Function Solutions

In addition to the rational solutions, P_{IV} admits solutions that are described by combinations of special functions. In particular, P_{IV} has solutions expressible in terms of combinations of parabolic cylinder functions, $D_\nu(\zeta)$ (Ref. 1, Chapter 12), or confluent hypergeometric functions, ${}_1F_1(a, b; \zeta)$ (Ref. 1, Chapter 13).

There are three distinct determinantal representations of solutions in terms of the parabolic cylinder functions available^{12,13}; however, only one of these expressions has been confirmed numerically.¹⁰ There are still other, simpler expressions involving parabolic cylin-

Parabolic Cylinder Function Type Solutions, $u_{\nu,\epsilon,d_1,d_2}^{[PC;k]}{}_{12,13}$				
k	α	β	Special Choice	Special Solution
1	$-\epsilon(\nu + 1)$	$-2\nu^2$	$\nu \in \mathbb{Z}^+$ $\nu = 0$	Standard Hermite Complementary Error
2	$-\epsilon\nu$	$-2(\nu + 1)^2$	$\nu \in \mathbb{Z}^+$	Standard Hermite
3	$-\epsilon(2n - \nu)$	$-2(\nu + 1)^2$		

TABLE III. Parameter choices leading to solutions of P_{IV} expressible in terms of parabolic cylinder functions. Notice that there are two parameters d_1 and d_2 that can one can vary to generate a family of special function solutions for a given choice of α and β .

der functions that have been validated with the appropriate parameter choices for these appearing in table III. Also, figure 1 displays the locations of all of these parameter choices that have been discussed in the literature as black curves.

It should be noted that when $\epsilon = -1$ the solutions $u_{\nu,-1,d_1,d_2}^{[PC;k]}$, $k = 1, 2, 3$, are no longer always real along the real axis. Otherwise in this paper, only solutions of P_{IV} that are real along the real axis will be considered.

Further, in the case of these parabolic cylinder function solutions the parameters d_1 and d_2 can be combined into a single parameter d . This combination leads to a one parameter family of solutions for each fixed α and β satisfying the expressions in table III.

Particular choices of $\nu \in \mathbb{R}$ allow the solutions $u_{\nu,\epsilon,d_1,d_2}^{[PC;1]}$ and $u_{\nu,\epsilon,d_1,d_2}^{[PC;2]}$ to be expressed in special forms. For instance, if $\nu \in \mathbb{Z}^+$ the special function solutions reduce to standard Hermite polynomials, while if $\nu = 0$ then the solutions of P_{IV} can be expressed as complementary error functions^{6,14}.

More recently, it was discovered in the context of supersymmetric quantum mechanics that P_{IV} has solutions that can be described by confluent hypergeometric functions^{14,15}. The parameter choices corresponding to these solutions are actually a subset of the larger set of parameter choices that lead to solutions expressed in terms of parabolic cylinder functions. Let

$$v_0(z) = e^{-\frac{1}{2}z^2} \left({}_1F_1 \left(-\frac{1}{2}\nu, \frac{1}{2}; z^2 \right) + 2z \frac{\Gamma(-\frac{1}{2}\nu + \frac{1}{2})}{\Gamma(-\frac{1}{2}\nu)} (c_1 + ic_2) {}_1F_1 \left(-\frac{1}{2}\nu + \frac{1}{2}, \frac{3}{2}; z^2 \right) \right),$$

where $n \in \mathbb{Z}^+$, $\nu, c_1, c_2 \in \mathbb{R}$ and ${}_1F_1$ is the confluent hypergeometric function¹. Further,

define for $j = 1, 2, \dots$,

$$v_j(z) = \frac{1}{\sqrt{2}} \left(\frac{d}{dz} v_{j-1}(z) + z v_{j-1}(z) \right).$$

P_{IV} has solutions

$$u_{\nu, n, c_1, c_2}^{[CH;1]} = u(z; 2n - \nu, -2(\nu + 1)^2) = -z - \frac{d}{dz} \ln \left(\frac{\mathcal{W}(v_0(z), v_1(z), \dots, v_{n-1}(z))}{\mathcal{W}(v_0(z), v_1(z), \dots, v_n(z))} \right),$$

$n > 0$ with $\mathcal{W}$ the usual Wronskian¹⁴. When $n = 0$ this solution reduces to

$$u_{\nu, 0, c_1, c_2}^{[CH;1]} = u(z; -\nu, -2(\nu + 1)^2) = -z + \frac{d}{dz} \ln(v_0(z)),$$

Notice that the choice of nonzero c_2 leads to solutions that are not real along the real axis. Therefore, $P_{IV}(2n - \nu, -2(\nu + 1)^2)$ has a one parameter family of solutions when $c_2 = 0$ for each fixed value of ν and n .

If d_1 and d_2 are chosen to correspond to the choice of c_1 and c_2 , and vice versa, then the resulting solutions $u_{\nu, 1, d_1, d_2}^{[PC;2]}$ and $u_{\nu, 0, c_1, c_2}^{[CH;1]}$ are identical. In fact, the relationships are given by

$$\begin{aligned} d_1 &= 2\sqrt{2} - 2\sqrt{2}(c_1 + ic_2) \\ d_2 &= 2\sqrt{2} + 2\sqrt{2}(c_1 + ic_2). \end{aligned}$$

C. Asymptotic Approximation

Beyond the known closed form solutions, it is noted in Ref. 1, Section 32.11, that when $\alpha \in \mathbb{R}$ and $\beta = 0$, nontrivial solutions satisfying

$$u(z) \rightarrow 0, \text{ as } z \rightarrow +\infty \tag{11}$$

are asymptotic to

$$k \left(D_{\frac{1}{2}\alpha - \frac{1}{2}}(\sqrt{2}z) \right)^2 \text{ as } z \rightarrow +\infty \text{ and } k \neq 0, \tag{12}$$

where $D_\nu(\zeta)$ is, again, the parabolic cylinder function. A more detailed explanation of these asymptotics can be found in Ref. 1, Section 32.11, including connection formulae and behaviors as $z \rightarrow -\infty$.

When assuming the derivative terms in (1) are negligible, the method of dominant balance (see, e.g., Ref. 16, Section 3.4) leads to the quartic equation

$$\frac{3}{2}w(z)^4 + 4zw(z)^3 + 2(z^2 - \alpha)w(z)^2 + \beta = 0. \quad (13)$$

The roots of (13) supply asymptotic approximations as $z \rightarrow \pm\infty$, $z \in \mathbb{R}$, and any choice of α and β . Asymptotic expansion as $z \rightarrow +\infty$ reveals that for all α and β

$$w_{+1}^+(z; \alpha, \beta) = \frac{\sqrt{-2\beta}}{2z} + \frac{\alpha\sqrt{-2\beta} + 2\beta}{4z^3} + O\left(\frac{1}{z^5}\right) \quad (14)$$

$$w_{-1}^+(z; \alpha, \beta) = -\frac{\sqrt{-2\beta}}{2z} + \frac{-\alpha\sqrt{-2\beta} + 2\beta}{4z^3} + O\left(\frac{1}{z^5}\right) \quad (15)$$

$$w_{+1}^-(z; \alpha, \beta) = -\frac{2}{3}z + \frac{\alpha}{z} - \frac{2(2\alpha^2 + 3\beta)}{8z^3} + O\left(\frac{1}{z^5}\right) \quad (16)$$

$$w_{-1}^-(z; \alpha, \beta) = -2z - \frac{\alpha}{z} + \frac{6\alpha^2 + \beta}{8z^3} + O\left(\frac{1}{z^5}\right). \quad (17)$$

No other smooth asymptotic behaviors were observed in the numerical explorations. With the assumption $u(z) \in \mathbb{R}$ for $z \in \mathbb{R}$, only the latter two roots are available as asymptotic approximations when $\beta > 0$. Later in this paper, ICs leading to solutions asymptotic to (14)-(17) will be marked in several figures, described as pole counting diagrams, as shown in figure 3.

As with the information presented in this section, the rest of this paper will discuss only the asymptotic behaviors as $z \rightarrow +\infty$, $z \in \mathbb{R}$, since the symmetry (2) makes it clear that there are analogous solutions with similar asymptotic behaviors as $z \rightarrow -\infty$. This is seen by comparing the left and right frames in figure 2.

D. The Parameter Space and the Weyl Chambers

Based on the various symmetries, solution hierarchies and known closed form solutions, the parameter space of P_{IV} with $\beta \leq 0$ can be described in terms of the so-called Weyl chambers (see e.g., Refs. 17, Section II-A, 6, Section 26). These chambers feature a complete regularity in the $(\alpha, \sqrt{-2\beta})$ -plane, where α and β are the two free parameters in the P_{IV} equation (1).

All of the (α, β) pairs described in the literature leading to rational and special func-

tion solutions are shown in figure 1. First, the dark (blue)/light (yellow) hexagrams indicate the parameter values that admit instances of solutions described by Generalized Hermite/Okamoto polynomials. Next, the parameter choices along the black lines admit special function solutions that are described by combinations of either parabolic cylinder functions or confluent hypergeometric and gamma functions. Finally, for parameter choices along the line $\sqrt{-2\beta} = 0$ (i.e. $\beta = 0$) the literature contains asymptotic approximations along the real axis. Each of these cases is considered in one of the following sections.

The significance of the Weyl Chambers, when extended to complex α and β , is that a single chamber in theory provides all of the information to construct solutions for every arbitrary (α, β) pair. Gromak, et al, state in Ref. 6, Section 25, “To construct the solutions of (1) for arbitrary values of parameters (α, β) it is sufficient to construct solutions for every (α, β) in the domain

$$\mathcal{F} := \left\{ (\alpha, \beta) \mid 0 \leq \text{Re}(\alpha) \leq 1, \text{Re}(\sqrt{-2\beta}) \geq 0, \text{Re}(\sqrt{-2\beta} + 2\alpha) \leq 2 \right\}.” \quad (18)$$

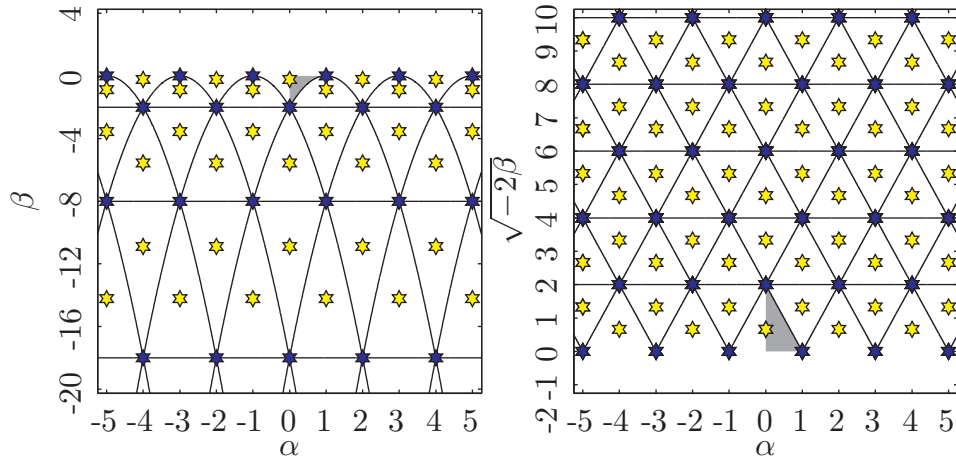


FIG. 1. Two views of the Weyl Chambers. The shaded region indicates the real part of the fundamental domain given in (18). Both figures show several of the chambers and locations of the rational and special function solutions to P_{IV} (dark hexagrams (blue) represent generalized Hermite type, light hexagrams (yellow) show generalized Okamoto type, and lines (black) show parabolic cylinder and confluent hypergeometric types).

Returning to $\alpha, \beta \in \mathbb{R}$ the region $\mathcal{F}$ is indicated by the shaded region in each subplot of figure 1. Notice that every parameter choice leading to a rational or special function solution of P_{IV} has $\beta \leq 0$. This is also true for the asymptotic approximation (12). For this reason, part of this study will be devoted to the mostly unexplored region of $\beta > 0$.

IV. THE NUMERICAL METHOD AND EXPLORATION APPROACH

Explorations of the vast space of parameters and ICs require a fast numerical method and a systematic approach for comparing solutions of different parameter choices. These techniques are discussed here.

The extensive pole fields appearing in these solutions have motivated the development of various solution techniques over the years since their discovery. However, many of the previous methods were limited in the choice of the parameters in the coefficients by considering special forms of the equation (e.g. Riemann Hilbert problems¹⁸), constrained to the real axis^{19,20}, or restricted to a small domain around the origin²¹. The presently used method extends the ‘pole vaulting’ idea¹⁹ in three fundamental ways: (i) use of a ‘pole friendly’ ODE integrator³, (ii) not using any rigid choices of diversion paths around a pole, but instead utilizing a freely branching network of paths in the complex plane, and (iii) targeting paths toward whole regions in the complex plane (rather than only toward other real axis locations). A survey of many of these existing numerical methods appears in Ref. 22.

A. A Brief Description of the Numerical Method

The numerical scheme introduced in Ref. 2 features very high orders of accuracy (typically 30 to 50), minimal loss of accuracy in the vicinity of poles, and a flexible path selection strategy that can efficiently cover large areas of the complex plane, while allowing arbitrary values of α and β . When integrating from one start location to a single end location this scheme uses the following strategy, which will be called pole avoidance:

1. Choose the location of the initial condition as the first expansion point.
2. Compute the Padé approximation about the expansion point.
3. Evaluate the Padé approximation a distance h away in each of five directions in a swath directed toward the target point and choose as the next expansion point the one with the smallest solution magnitude.
4. Unless the target point has been reached, return to step 2.

This pole avoidance strategy is effective when finding the solution to an IVP at a single point. However, if the solution is desired at many different points (for instance, for the

visualization of the solution over a region in the complex plane) the method is extended to the pole field solver.

1. Set up a coarse grid of target points in the complex plane.
2. Select the target points in random order.
3. Apply the pole avoidance strategy to reach a predetermined neighborhood of the current target point, starting from the closest point that has already been evaluated. In the first step this is the location of the IC.
4. Once all of the coarse grid target points have been accounted for, set up a fine grid of the desired evaluation points.
5. Compute a single last step from the end of each of the previous paths to several nearby fine grid evaluation points.

B. Pole Counting

The pole field solver makes it possible to rapidly view solutions for a variety of initial conditions. Therefore, to explore the differences in solution characteristics for each fixed choice of α and β , but for varying $(u(0), u'(0)) \in \mathbb{R}^2$, the number of poles on either the positive or negative real axis is examined. This, paired with the asymptotic behavior discussed in section III C, allows the characterization of the numerous solution possibilities for each fixed α and β . Figure 2 (adapted from Ref. 5) provides a prototypical example in the case of $\alpha = \beta = 0$. This figure displays the number of poles on the positive and negative real axes for each choice of initial conditions shown, and each of the frames is dubbed a pole counting diagram.

Consider, for now, only the right frame in figure 2, since the left is completely analogous due to the symmetries discussed in section II A. Each of the ICs marked by a curve or contained within a shaded region generates a solution with a finite number of poles on the positive real axis. The color bar indicates the exact number of poles for a given initial condition with darker and lighter indicating odd and even numbers of poles, respectively. On the other hand, ICs neither contained in a shaded region nor marked by a curve should

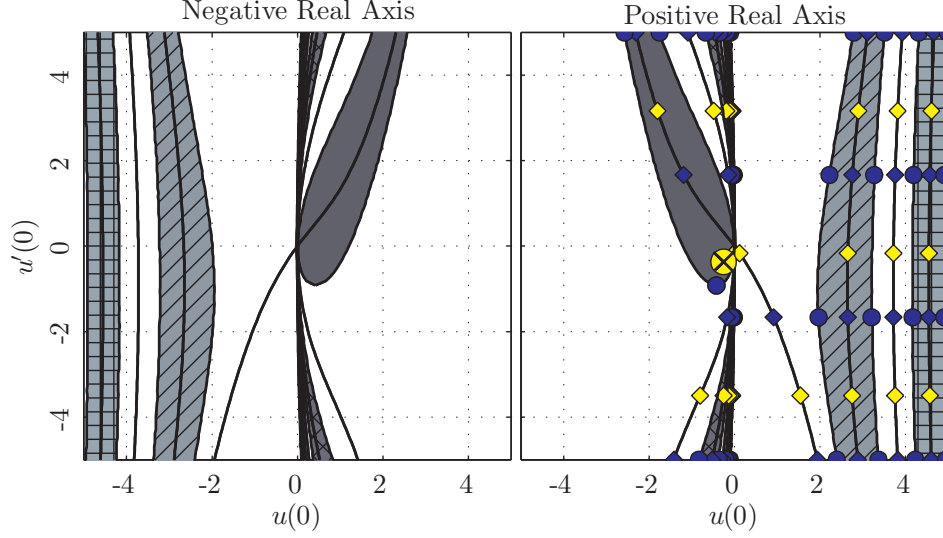


FIG. 2. Number of poles on the positive and negative real axes for $\alpha = 0$ and $\beta = 0$. For a description of the markers and shading see figure 3. When $\beta = 0$ the ICs marked with light and dark diamonds are precisely those satisfying the decaying asymptotic condition (12).

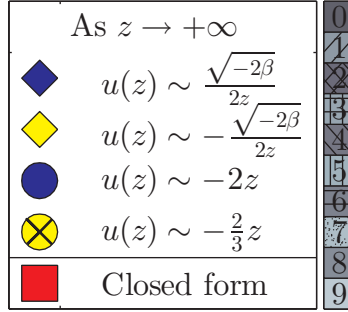


FIG. 3. Legend and color bar for figures 2, 10, 13, 14, 15, 16, 17, 18, 19, and 20. The legend shows the markers indicating the ICs that generate the dominant asymptotic behaviors (14)-(17) and closed form solutions. If a marker occurs on a curve, then the dominant behavior or type of closed form solution occurs for all of the ICs along that curve. If a marker is emphasized by containing an “ $\times$ ”, then it indicates an isolated IC matching the dominant behavior or the IC generates an isolated rational solution. The gray-scale/pattern bar on the right indicates the number of poles on the positive or negative real axis.

generate solutions with an infinity of poles on the corresponding half (positive/negative) of the real axis.

In this case of $\alpha = \beta = 0$, each of the shaded regions in the right half-plane contains ICs that generate solutions with an odd number of poles on the positive real axis, while the $u(0)$, $u'(0)$ values along the isolated curves lead to solutions with an even number, with the opposite holding in the left half-plane.

Most of the ICs in the shaded regions generate solutions that oscillate as $z \rightarrow +\infty$ (note that an oscillation is simply a change in the sign of the derivative); however, each initial condition marked by a curve, located at the boundary of a shaded region, or designated by an isolated marker has no oscillations as $z \rightarrow +\infty$. These solutions are precisely those that are asymptotic to the roots of the quartic equation (13) as $z \rightarrow +\infty$. The appropriate root is indicated by the symbols shown in left frame of figure 3. In the case of $\alpha = \beta = 0$ (generally, when $\beta = 0$) the solutions matching the behaviors w_μ^+ , $\mu = \pm 1$, are the solutions that satisfy the decaying asymptotic condition (12). When two markers appear along the same curve, those ICs generate solutions matching both behaviors (in separate intervals of the real axis), as shown in, for example, figures 11, 12, 26, and 27 (see section VID for further discussion).

C. Confirmation of Solution Transformations Using the Numerical Method

The numerical explorations in this study begin with confirmation of the transformations (5)-(7). This confirmation was completed by first computing the exact transformations of a (numerically obtained) solution using (5)-(7) at each point in the solution. Then, the transformed results were compared to numerical solutions generated using a single transformed initial condition.

It was noted⁵ that $P_{IV}(0,0)$ has a solution with a pole-free half-plane. Figure 4 shows a counterpart to this solution with $\alpha = 0.5$ and $\beta = -0.5$. The transformations $u_{k,\mu}^\pm$ lead to the solutions of P_{IV} in figures 6 and 7. In the left frame of figure 4 and all of the frames in figure 6, the zeros are marked with “ $\times$ ” (red) while the poles are marked with circles (blue and yellow for residues of +1 and -1, respectively). This same convention will be used for pole locations throughout the rest of the paper, but zeros will not always be shown since they appear very regularly with the poles. The left frame of figure 4 shows a pole field in each of the upper- and lower-left quadrants of the complex plane, while the right frame indicates that this solution approaches roughly $-2/3z$ as $z \rightarrow +\infty$ and $-2z$ as $z \rightarrow -\infty$ for $z \in \mathbb{R}$ with a zero of order 1 at $z \approx 0.75$.

Notice that in figure 6 the general locations of the pole fields in the upper- and lower-right quadrants are maintained; however, aside from this similarity, it is cumbersome to characterize how the transformations (5) through (7) alter the locations of these poles and

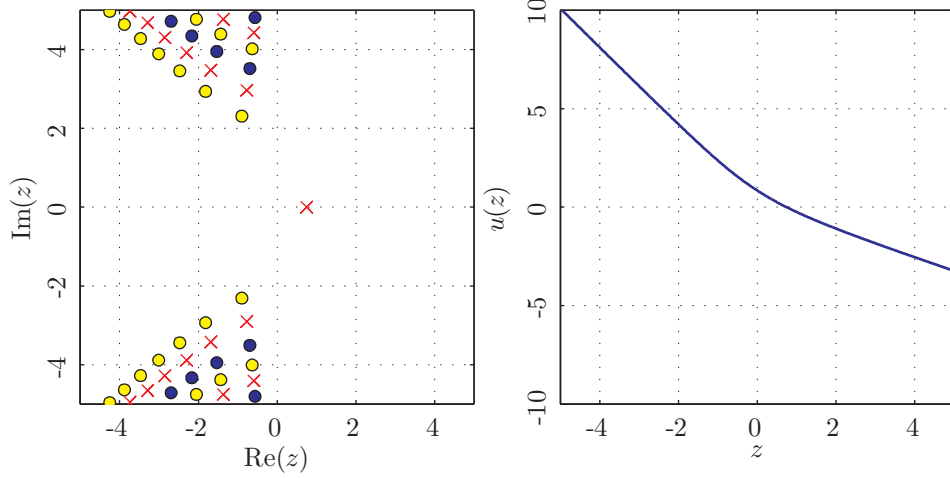


FIG. 4. Solution with a pole free half-plane for $\alpha = 0.5$ and $\beta = -0.5$. The left frame shows the zero and pole locations, while the right shows the solution along the real axis.

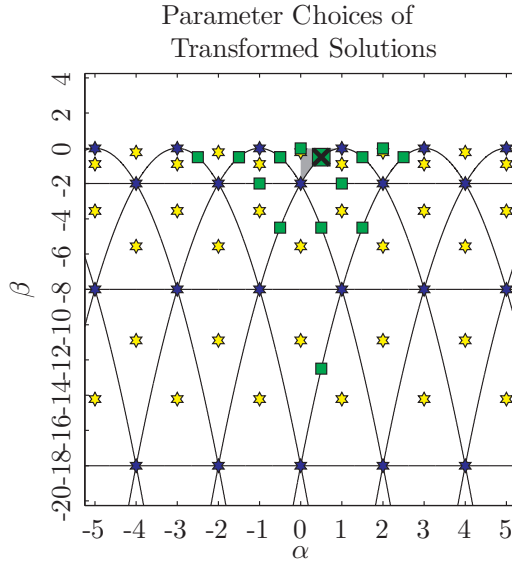


FIG. 5. Another view of the Weyl chamber. The larger square with an $\times$ marks the original choice of $\alpha = 0.5$ and $\beta = -0.5$. The other squares mark the parameter space locations of the transformed solutions.

zeros. Even for a fixed k , the transformations $u_{k,\mu}^\pm$ can vary drastically for the choice of μ and the upper and lower sign. Further, the transformations (5) and (6) suggest at first glance that any zeros of $u(z)$ should be poles of $u_{k,\mu}^\pm(u(z), z)$, $k = 1, 2$, and $u_4^\pm(u(z), z)$; however, this simple analysis does not tell the whole story, and is certainly not always the case.

Consider the solutions to P_{IV} asymptotic to the roots of the quartic equation (13) as $z \rightarrow +\infty$. Table IV contains the resulting asymptotic behaviors. Specifically, if $u(z)$ possesses the

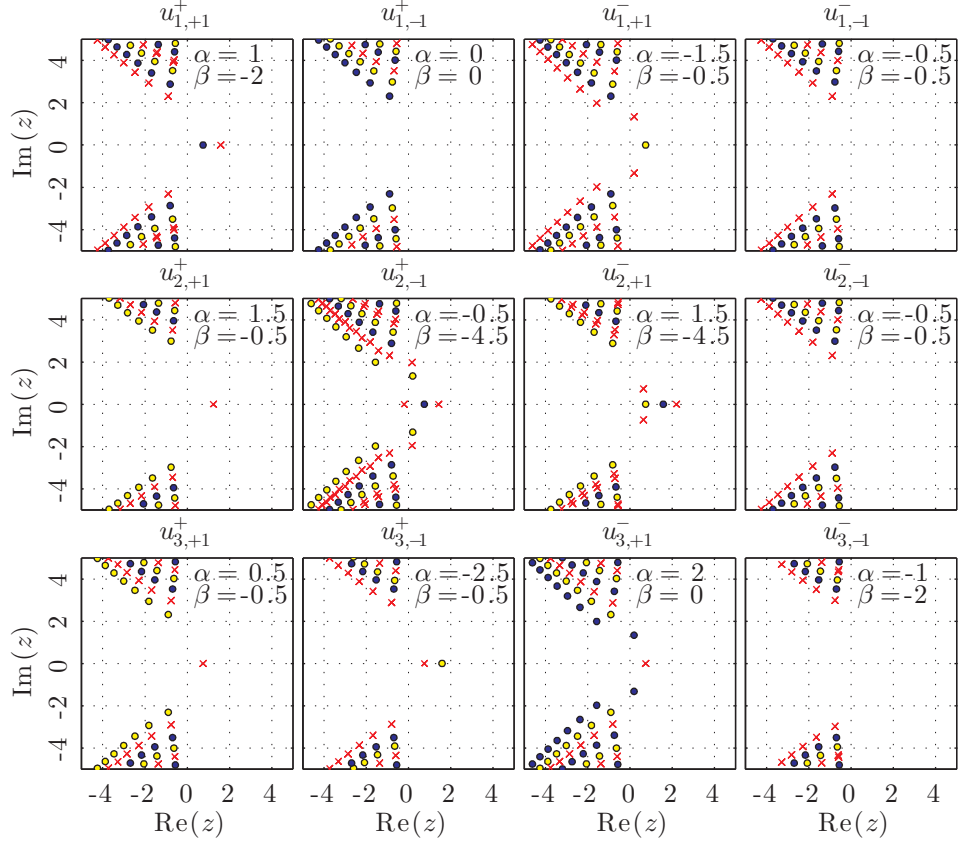


FIG. 6. Zero and pole locations of solutions to P_{IV} resulting from the applications of (5) through (7) to the solution figure 4.

asymptotic behavior in the row marked $u(z)$ as $z \rightarrow +\infty$ and $z \in \mathbb{R}$, then the transformed solutions possess the asymptotic behavior in the following rows as $z \rightarrow +\infty$ and $z \in \mathbb{R}$. A further discussion of the asymptotic behaviors $w_\mu^\pm$, with $\mu = \pm 1$, appears in section III C.

V. NUMERICAL ILLUSTRATIONS OF THE FUNDAMENTAL DOMAIN

Solution types occurring for parameter choices in the fundamental domain are discussed in the following sections. These (and subsequent) sections describe some solutions as having adjacent pole free sectors. This terminology arises from evidence in the numerical explorations that the poles in the solutions of P_{IV} align in the eight sectors shown in figure 8. Further discussions of these sectors are available⁵.

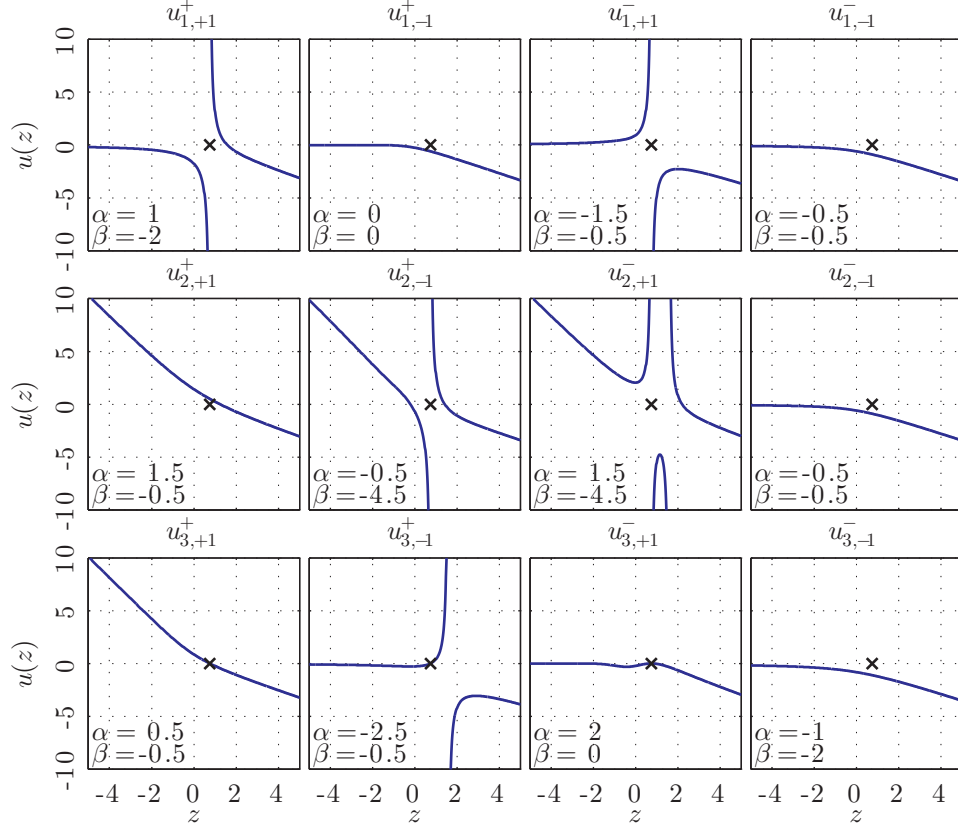


FIG. 7. Solutions along the real axis to P_{IV} resulting from the applications of (5) through (7) to the solution figure 4.

Leading Order Asymptotic Behavior of $u(z)$ as $z \rightarrow +\infty$				
$u(z)$	$w_{+1}^+ \sim \frac{\sqrt{-2\beta}}{2z}$	$w_{-1}^+ \sim -\frac{\sqrt{-2\beta}}{2z}$	$w_{+1}^- \sim -\frac{2}{3}z$	$w_{-1}^- \sim -2z$
Leading Order Asymptotic Behavior of Transformed Solutions as $z \rightarrow +\infty$				
$u_{1,\mu}^+(u(z), z)$	$-2z$	$\frac{-2\mu-2\alpha-\sqrt{-2\beta}}{4z}$	$-\frac{2}{3}z$	$\frac{2\mu+2\alpha+\sqrt{-2\beta}}{4z}$
$u_{1,\mu}^-(u(z), z)$	$\frac{2\mu-2\alpha+\sqrt{-2\beta}}{4z}$	$-2z$	$-\frac{2}{3}z$	$\frac{-2\mu+2\alpha+\sqrt{-2\beta}}{4z}$
$u_{2,\mu}^+(u(z), z)$	$\frac{\mu-2\alpha-(2-\alpha)\sqrt{-2\beta}+\beta\mu}{(2+2\alpha\mu-\mu\sqrt{-2\beta})z}$	$\frac{2\mu-\sqrt{-2\beta}}{2z}$	$-\frac{2}{3}z$	$-2z$
$u_{2,\mu}^-(u(z), z)$	$\frac{2\mu+\sqrt{-2\beta}}{2z}$	$\frac{1-2\alpha\mu-\mu(2+\mu\alpha)\sqrt{-2\beta}+\beta}{(2\mu+2\alpha+\sqrt{-2\beta})z}$	$-\frac{2}{3}z$	$-2z$
$u_{3,\mu}^+(u(z), z)$	$\frac{2\mu-2\alpha+\sqrt{-2\beta}}{4z}$	$-2z$	$-\frac{2}{3}z$	$\frac{-4\mu\alpha+2\alpha^2+\beta}{(-4\mu+4\alpha+2\sqrt{-2\beta})z}$
$u_{3,\mu}^-(u(z), z)$	$-2z$	$\frac{2\mu-2\alpha-\sqrt{-2\beta}}{4z}$	$-\frac{2}{3}z$	$\frac{4\alpha-2\mu\alpha^2-\mu\beta}{(4-4\mu\alpha+2\mu\sqrt{-2\beta})z}$

TABLE IV. Asymptotic behaviors of transformed solutions. If $u(z)$ possesses the asymptotic behavior in the row marked $u(z)$ as $z \rightarrow +\infty$ and $z \in \mathbb{R}$, then the transformed solutions possess the asymptotic behavior in the following rows as $z \rightarrow +\infty$ and $z \in \mathbb{R}$. With restriction to the solutions that are real on the real axis, all options in the table are feasible when $\beta \leq 0$. When $\beta > 0$ those that contain the term $\sqrt{-2\beta}$ are not.

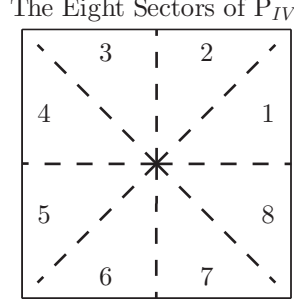


FIG. 8. The eight sectors of poles in the solutions of P_{IV} .

A. An Exploration of the Fundamental Domain

In section IIID the fundamental domain (18) was introduced, and it was noted that solutions for all parameter choices in theory can be found by applying the transformations (5) through (7) to the solutions in this domain. However, the literature describes solutions in this domain only for the cases $\alpha = \beta = 0$ (numerical and asymptotic solutions), ($\alpha = 0, \beta = -2/9$) (a rational solution), along the line $\beta = 0$ (asymptotically decaying solutions), and along the curve $\beta = -2(\alpha - 1)^2$ (asymptotic, rational and special function solutions). All of these occur on the boundaries of the fundamental domain. In particular, the special solutions described for each of these parameter choices are indicated in figure 9.

1. *Parameter Choices with Rational or Special Function Solutions*

It should again be noted that, for each of the parameter choices ($\alpha = 0, \beta = -\frac{2}{9}$) and along the curve $\beta = -2(\alpha - 1)^2$, the closed form or asymptotic solutions only lead to a single solution or a one parameter family of solutions in the $u(0)$ versus $u'(0)$ plane. To gain some insight into arbitrary ICs (in the same manner as figure 2) the frames in figure 10 show the number of poles appearing on the positive real axis for each of the two remaining vertices of the fundamental domain, as well as the case ($\alpha = 0, \beta = -2/9$). A detailed description of the markers and shading is given in figure 3.

Within the frames of figures 2 and 10 it is easy to see that the ICs of solutions asymptotic to the roots of (13) appear regularly as the boundaries of shaded regions or along curves generated by the ICs of solutions asymptotic to (12) or those of special function solutions. To this point, the last two frames show a peculiar behavior of these asymptotic solutions

Closed Form and Asymptotic Solution
Types of the Fundamental Domain

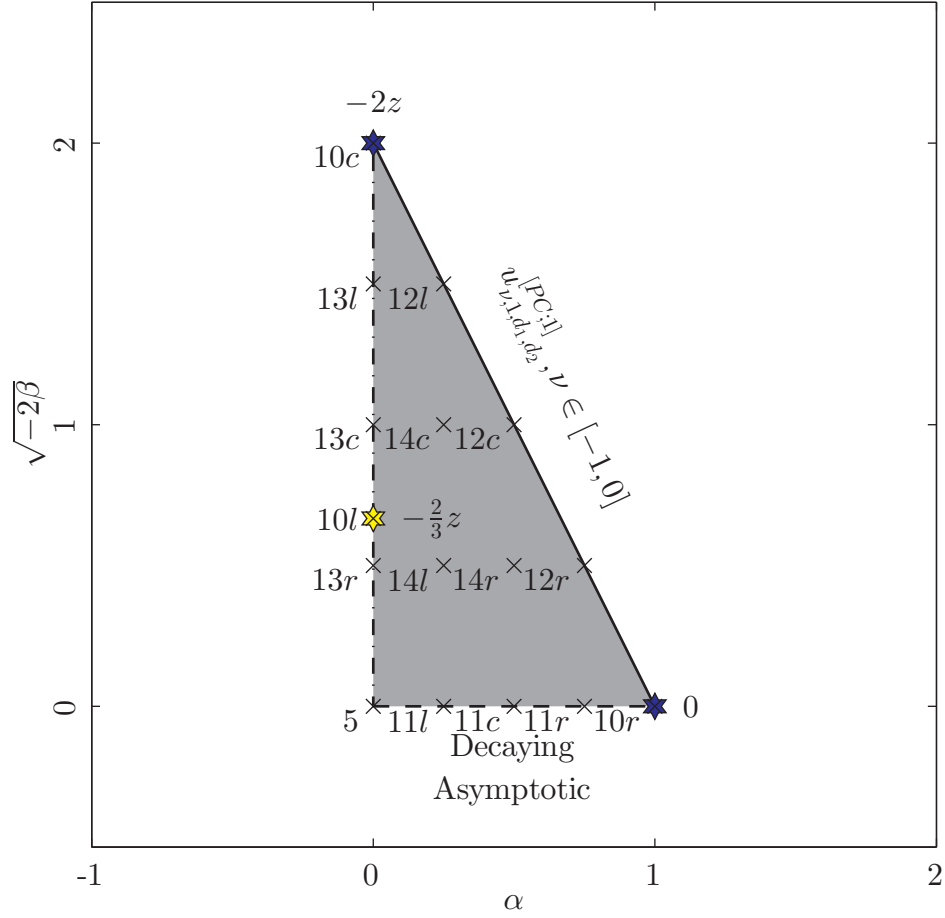


FIG. 9. Locations of the closed form and asymptotic solution types appearing in the fundamental domain. The $\times$'s mark the parameter choices in the fundamental domain where pole counts will be shown along with the appropriate figure number (l , c , and r refer to the left, center, and right frames, respectively).

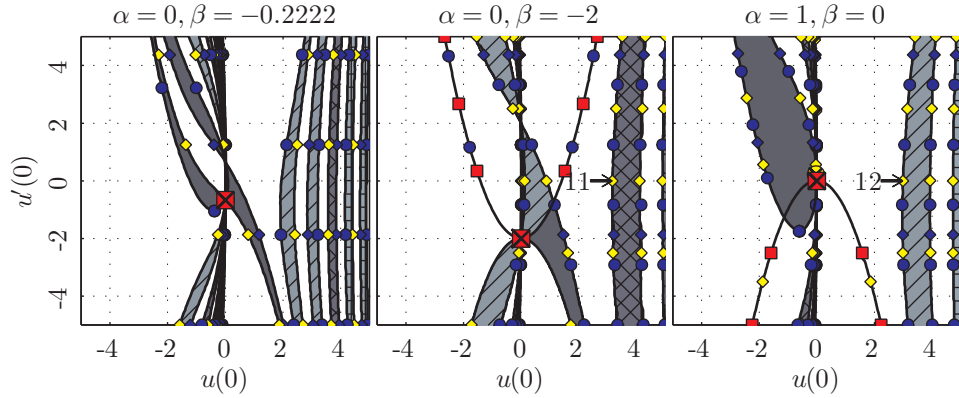


FIG. 10. Number of poles on the positive real axis for $\alpha = 0$ and $\beta = -2/9$, $\alpha = 0$ and $\beta = -2$, and $\alpha = 1$ and $\beta = 0$. A detailed description of the markers and shading is given in figure 3.

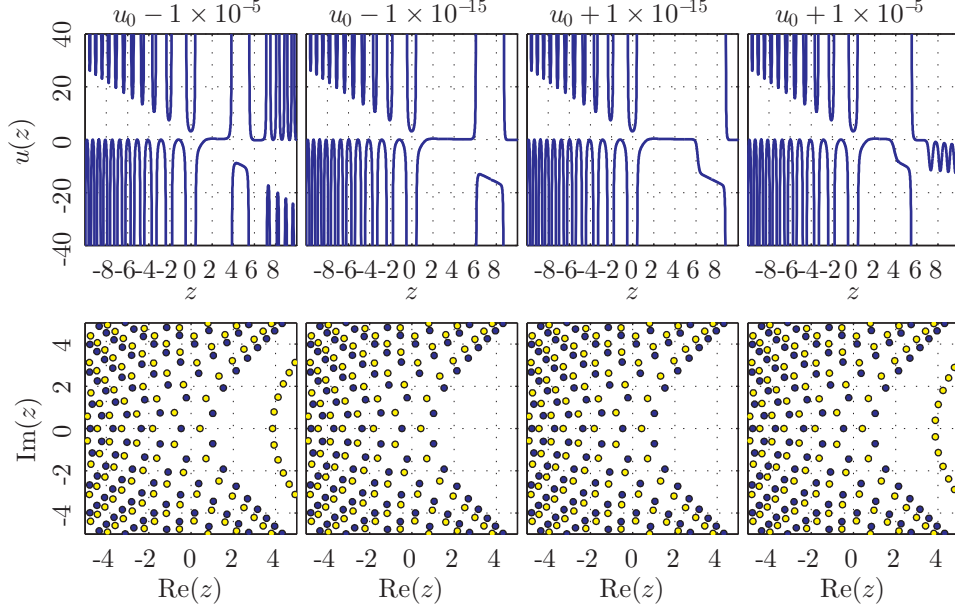


FIG. 11. Solutions (along the real axis (top) and pole locations and residues (bottom)) with adjacent pole free sectors for $\alpha = 0$ and $\beta = -2$. $u'(0) = 0$ and $u_0 = 3.170110354518507$. This initial condition is marked with an arrow in the center frame of figure 10.

when the α and β choices occur at the vertex of a Weyl chamber. For these solutions, the behaviors of w_μ^+ , $\mu = \pm 1$, and w_{-1}^- are present in the same solution, but in different segments of the positive real axis. Take, for instance, the ICs for $(\alpha = 0, \beta = -2)$ and $(\alpha = 1, \beta = 0)$ indicated by the arrows in the second two frames of figure 10. Along the curves containing these ICs there are two or three separate markers. The solutions in a neighborhood of these particular ICs are shown in figures 11 and 12, illustrating that different dominant asymptotic behaviors can occur in the same solution (but, in different segments of the real axis).

2. *Parameter Choices Along the Boundary* $\beta = 0$

When the boundary $\beta = 0$ is considered the literature generally only describes solutions to P_{IV} that decay asymptotically as $z \rightarrow +\infty$. Considering the frame in the right of figure 2, all of the frames of figure 13, and the rightmost frame of figure 10, the ICs generating these solutions appear as curves with the appropriate markers (i.e. those shown in figure 3). These ICs are precisely the ones that correspond to solutions matching both the roots w_μ^+ , $\mu = \pm 1$. That is, these trends are both present when $\beta = 0$.

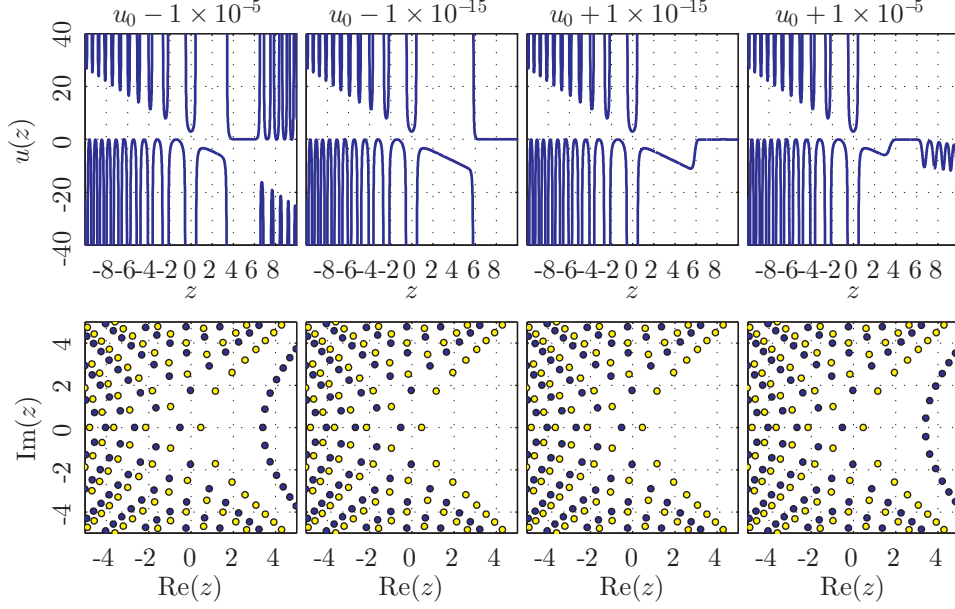


FIG. 12. Solutions (along the real axis (top) and pole locations and residues (bottom)) with adjacent pole free sectors for $\alpha = 1$ and $\beta = 0$. $u'(0) = 0$ and $u_0 = 2.989670219313871$. This initial condition is marked with an arrow in the right frame of figure 10.

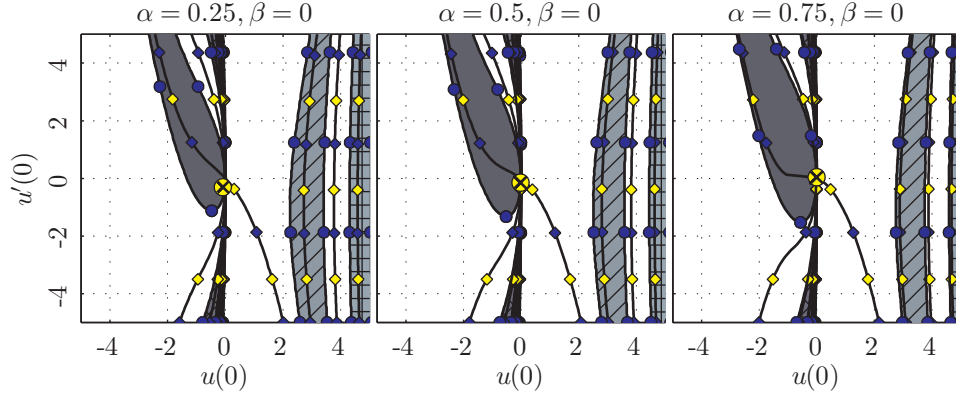


FIG. 13. Number of poles on the positive real axis for parameter choices on the boundary of the fundamental domain where $\beta = 0$. A detailed description of the markers and shading is given in figure 3. Here the light and dark diamonds refer to the solutions that match the behaviors w_μ^+ , $\mu = \pm 1$. These ICs are precisely those satisfying the decaying asymptotic condition (12).

3. Parameter Choices Along the Boundary $\beta = -2(\alpha - 1)^2$

Next, P_{IV} has a one-parameter family of solutions expressible in terms of the parabolic cylinder function or confluent hypergeometric function for each choice of α and β along the boundary described by $\beta = -2(\alpha - 1)^2$. Table III gives two choices of the parameters leading to these types of solutions. It was noted previously that the choice of $\epsilon = -1$ leads

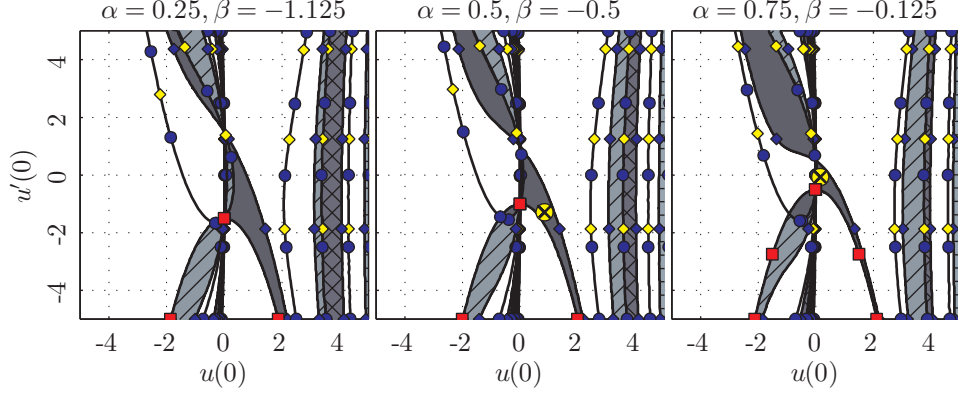


FIG. 14. Number of poles on the positive real axis for parameter choices on the boundary of the fundamental domain where $\beta = -2(\alpha - 1)^2$. A detailed description of the markers and shading is given in figure 3.

to solutions that are not always real along the real axis. Further, the parameter choices $\beta = -2(\alpha - 1)^2$ only satisfy the relationships for α and β (excluding $\epsilon = -1$) leading to the solutions $u_{\nu,1,d_1,d_2}^{[PC;k]}$ if $k = 2$. Therefore, only the initial conditions leading to solutions $u_{\nu,1,d_1,d_2}^{[PC;2]}$ are explicitly shown in figure 14. For these parameter choices the trends of $w_{-1}^{\pm}$ are present, and these are again $u(z) \rightarrow O\left(\frac{1}{z}\right)$ and $u(z) \rightarrow -2z$ as $z \rightarrow \infty$ and $z \in \mathbb{R}$.

4. Parameter Choices Along the Boundary $\alpha = 0$

Finally, along the boundary $\alpha = 0$ the locations of ICs generating solutions asymptotic to the roots w_{+1}^+ and w_{-1}^+ become distinct, separating or expanding into regions with a finite number of poles. This is easily seen in the sequence of frames in figure 15.

5. The Interior to the Fundamental Domain

Parameter choices interior to the fundamental domain behave much like those along the boundary $\alpha = 0$. In these cases, solutions asymptotic to each of the roots of (13) are generated from distinct ICs. This can be witnessed in figure 16.

6. A Note on Connection Formulae

Consider the left and right frames of figure 2, showing the number of poles along the negative and positive real axes, respectively. One finds that a segment of the curve extending

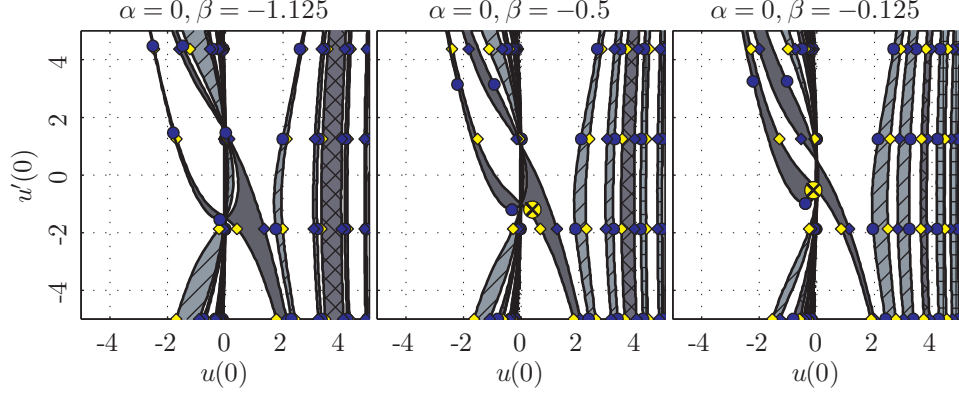


FIG. 15. Number of poles on the positive real axis for parameter choices on the boundary of the fundamental domain where $\alpha = 0$. Note that the initial conditions for the solution asymptotic to the root w_{-1}^- occur just outside the window shown here at $u(0) \approx 2.429702$ and $u'(0) \approx -7.568548$ (still within the same shaded region as the other two cases) in the case of $\beta = -1.125$. A detailed description of the markers and shading is given in figure 3.

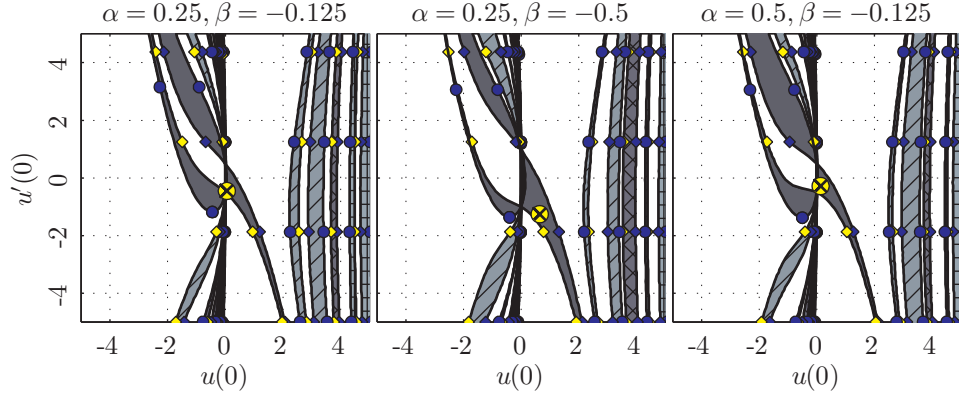


FIG. 16. Number of poles on the positive real axis for parameter choices interior to the fundamental domain. A detailed description of the markers and shading is given in figure 3.

from the origin and down to the right in the right frame cuts across the shaded region that extends from the origin up and to the right in the left frame. Along this segment P_{IV} therefore has solutions that are smooth in both directions. A similar analysis of the pole counting diagrams for any choice of α when $\beta = 0$ would result in an analogous family of solutions that are smooth in both directions. These appear to be the only examples of solutions that have connection formulae available in the literature (see, e.g. Refs. 1, 23, or 24).

Examination of figures 10 through 15 (together with the symmetry (2)) shows that similar comparisons of the number of poles on the positive and negative real axes will again identify

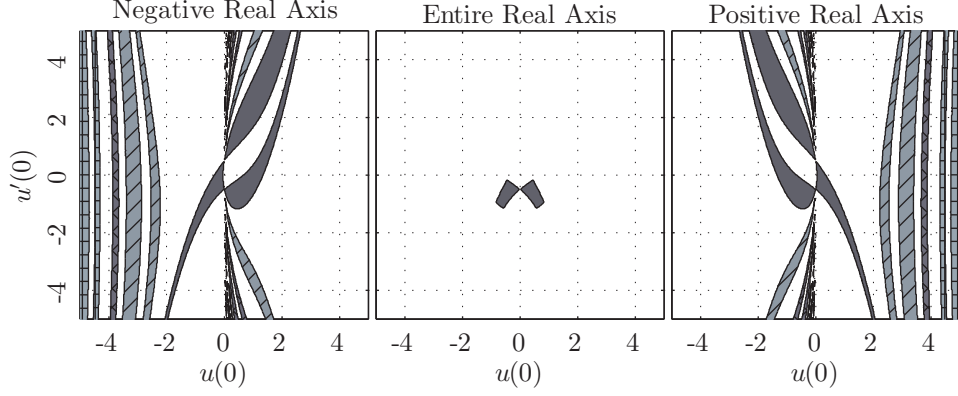


FIG. 17. Number of poles on the negative real axis (left), entire real axis (center), and positive real axis (right) for $\alpha = 0.25$ and $\beta = -0.125$.

solutions that are smooth in both directions for regions of ICs near $u(0) = u'(0) = 0$ in cases where β is negative. For instance, figures 15 and 16 indicate that such regions (sometimes only a curve) will exist for all parameter choices within the fundamental domain. Figure 17 illustrates this for a choice interior to the fundamental domain ($\alpha = 0.25$, $\beta = -0.125$).

In a following section, figure 20 will show that similar regions will also occur outside the fundamental domain when $\beta < 0$, however, with the difference that there now may be a finite number of poles on the real axis in either one or both directions. In contrast, positive choices of β do not seem to produce any such regions of ICs.

VI. SOLUTION PATTERNS OUTSIDE THE FUNDAMENTAL DOMAIN

The (α, β) space is far too wide to complete an exhaustive survey here. Therefore, the rest of this paper focuses on the unexplored space of $\beta > 0$ and highlights some solution types that seem to appear for all α and β .

A. The Unexplored Space of Positive Beta

Studies of P_{IV} with $\beta > 0$ are noticeably absent from the literature. For instance, all known closed form solutions occur only when β is nonpositive. Even the Bäcklund and Schlesinger transformations are only applicable to β -values that are nonpositive (assuming $u(z)$ is real when z is real). Exploration of such cases and knowledge of the tronquée like solutions that appear in the $\alpha = \beta = 0$ case suggests that solutions with $\beta > 0$ also feature

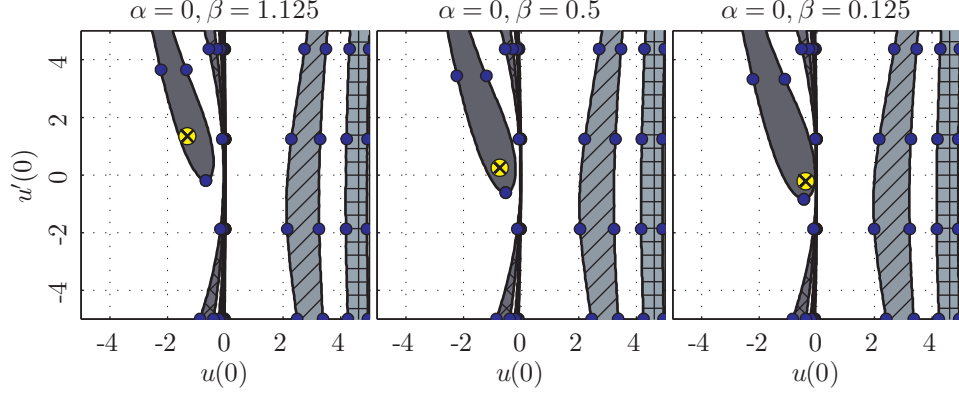


FIG. 18. Number of poles on the positive real axis for parameter choices where $\alpha = 0$ and $\beta > 0$. A detailed description of the markers and shading is given in figure 3.

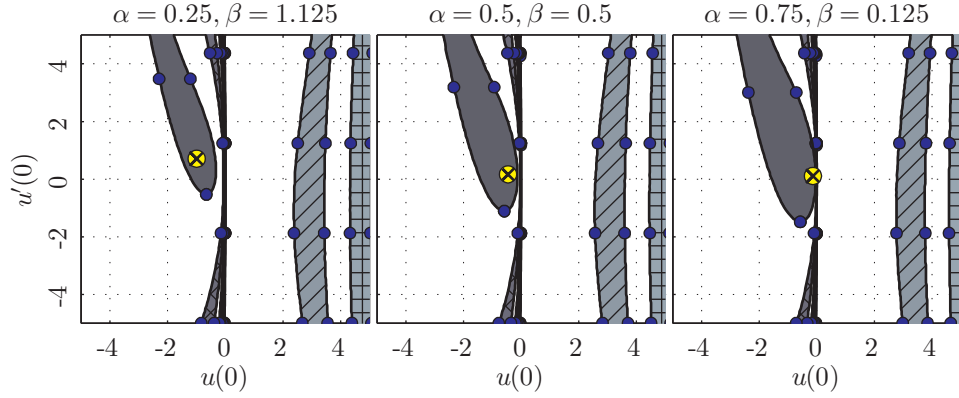


FIG. 19. Number of poles on the positive real axis for parameter choices where $\beta = 2(\alpha - 1)^2$ and $\beta > 0$. A detailed description of the markers and shading is given in figure 3.

noteworthy characteristics. For instance, there are further analogues to the solution that is pole free for a half-plane.

The asymptotic behaviors (14) and (15) no longer occur as solutions that are real along the real axis, due to the term $\sqrt{-2\beta}$. Therefore, the figures 18 and 19 are much simpler than their counterparts with a single IC generating the asymptotic behavior of $w_{+1}^{-1} \sim -2/3z$ and ICs along the boundaries of regions with finite poles leading to that of $w_{-1}^{-1} \sim -2z$.

B. Parameters Larger in Magnitude

This section illustrates some α, β choices slightly larger in magnitude. When $\beta > 0$ there is little difference from the choices presented in the earlier figures. However, nonpositive choices of β become far more complicated without indicating the existence of further types

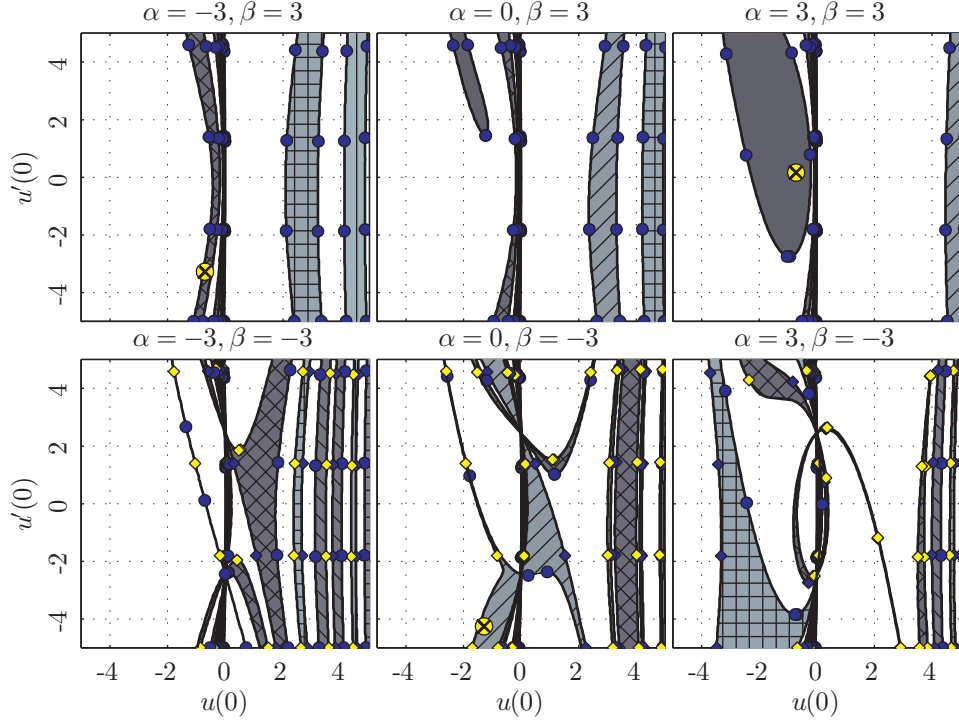


FIG. 20. Number of poles on the positive real axis along the edge of a grid exterior to the fundamental domain. The initial conditions for solutions asymptotic to w_{+1}^- in the top middle, bottom left, and bottom right frames occur outside of the domain shown at $(u(0) = -4.6822, u'(0) = 20.7787)$, $(u(0) = -10.7942, u'(0) = 120.3759)$, and $(u(0) = 49.4606, u'(0) = -2442.3215)$, respectively. The locations of these parameters in α vs. β space are shown later in figure 21. A detailed description of the markers and shading is given in figure 3.

of solutions with special characteristics. Even parameter choices in adjacent Weyl chambers generate significantly different behaviors near $u(0) = u'(0) = 0$.

C. Solutions With a Nearly Pole Free Half Plane

It was noted⁵ that when $u(z)$ satisfies the decaying asymptotic condition (12) and $\alpha = \beta = 0$ a particular choice of k leads to a solution that is pole free across the entire left half-plane. A similar solution is shown for $\alpha = 0.5$ and $\beta = -0.5$ in figure 4. Solutions with a nearly pole free half plane are not confined to only these special choices of α and β .

In fact, evidence suggests that for each α and β there exists at least one such solution, and very likely only one. The likelihood that there is only one such solution for each α and β pair makes this solution a prime candidate for comparing and making connections between all parameter choices.

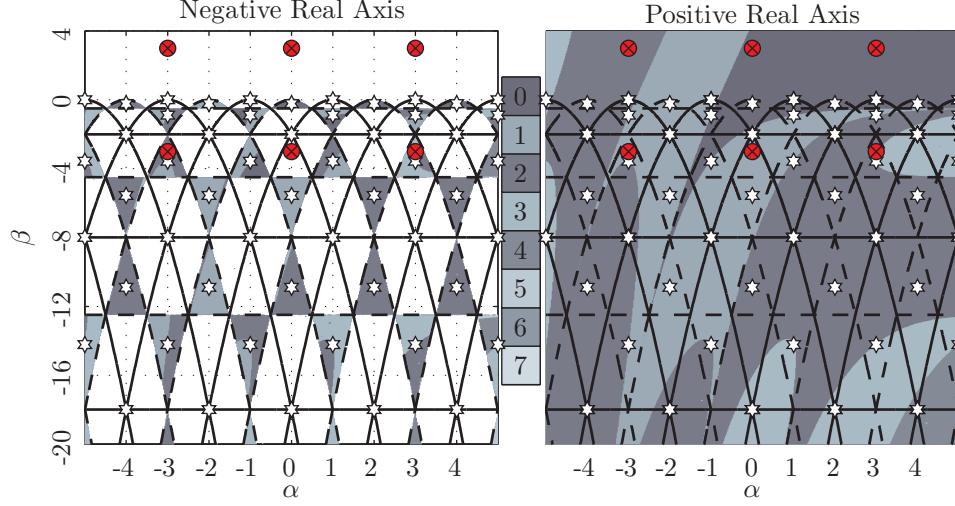


FIG. 21. Number of poles on the positive (right) and negative (left) real axis for solutions asymptotic to $w_{+1}^- \sim -2/3z$ as $z \rightarrow +\infty$ and $z \in \mathbb{R}$ and each α and β . The solid curves indicate the boundaries of the Weyl chambers, while the dashed lines show the boundaries of regions of finite poles on both the positive and negative real axes. Note that in this case $\beta > 0$ implies an infinity of poles along $\mathbb{R}^-$. The circles (red) containing an $\times$ indicate those parameters shown in figure 20. The changes in shading occur simultaneously in the left and right frames corresponding to a pole moving from one half of the real axis (positive/negative) to the other.

For each α and β this special solution type is asymptotic to the root $w_{+1}^- \sim -2/3z$ as $z \rightarrow +\infty$ and $z \in \mathbb{R}$. Knowing this, computing the initial conditions leading to such a solution is a simple matter of solving a boundary value problem (BVP). Applying the familiar methodology of counting poles along the positive, and now negative, real axes allows the identification of further special characteristics of these solutions.

In figure 21 the pole counts are shown along the negative and positive real axes (left and right frames, respectively) overlaid with the Weyl chambers marked by solid curves. Also in these frames, dashed lines mark the boundaries of regions in the α versus β plane where these solutions have only a finite number of poles on the negative real axis. Notice that these dashed curves form a regular structure similar to that of the Weyl chambers, with the parabolas offset by one unit on the α axis and the horizontal lines occurring at β values where these new parabolas and those from the Weyl chambers intersect.

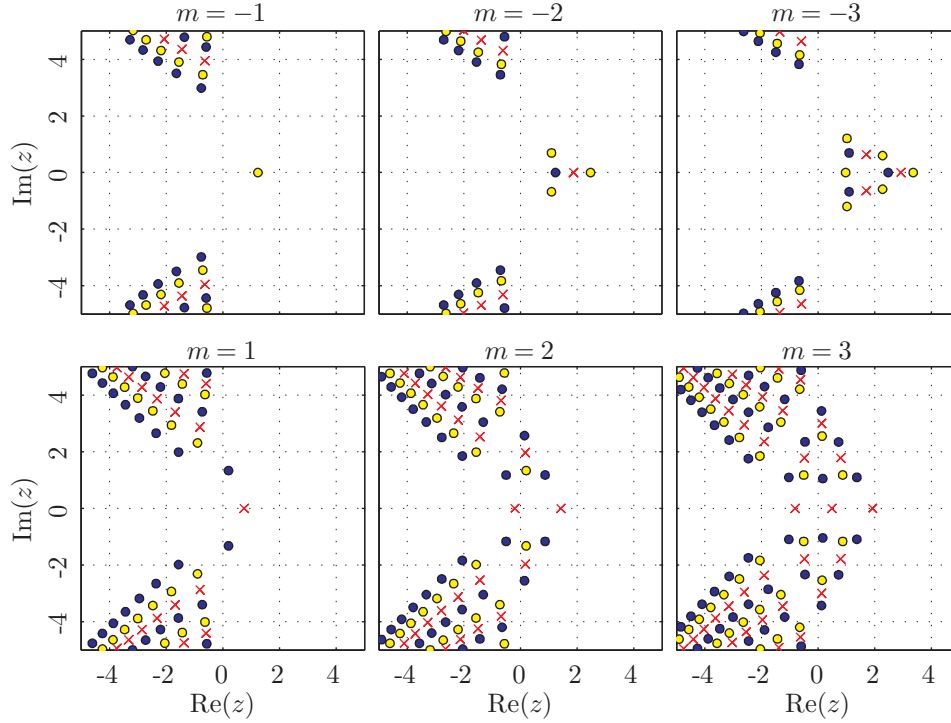


FIG. 22. Zero and pole locations of solutions to (1) with various values of m . Note that $\alpha = 2m$ and $\beta = 0$. The case $\alpha = m = 0$ is shown in one of the subplots of figure 5.

1. The Tops of the Parabolas

To begin, consider the parameter choices at the tops of these new parabolas. These occur at $\alpha = 2m$ and $\beta = 0$, $m \in \mathbb{Z}$. In these cases the poles nearest the origin form very regular patterns. Examples for several different choices of m are shown in figure 22. Notice the pole structure near the center of these figures. When $m < 0$ poles of residue +1 align in a structure similar to the roots with a positive real part of the degree m Okamoto I polynomial, while poles of residue -1 appear similar to the roots of the degree $m - 1$ polynomial. On the other hand, when $m > 0$ the poles of residue +1 (likewise, -1) align in a structure similar to all of the roots of the order $m + 1$ (likewise, m) polynomials. Note that the Okamoto I polynomials in this context are singly indexed as in Ref. 17 while those in the rational solutions of P_{IV} are doubly indexed generalized Okamoto polynomials as in Ref. 11.

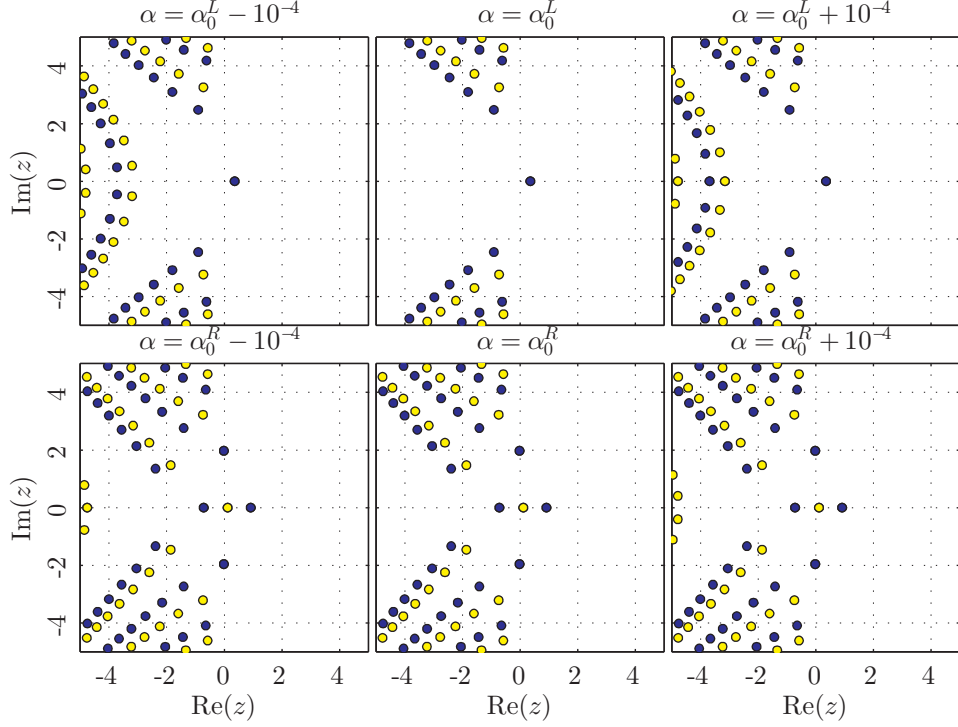


FIG. 23. Solutions (pole locations and residues) normal to the parabola $\beta = -2(\alpha - 2)^2$. All frames depict the solutions asymptotic to $-2/3z$ as $z \rightarrow +\infty$. The center frames occur directly along the parabolas where $\alpha = \alpha_0 = 1.25$ (top) and $\alpha = \alpha_0 = 2.75$ (bottom). The left and right frames in both the top and bottom then depict the solutions along the line normal to the parabola at $\alpha = \alpha_0$ at $\alpha_0 \pm 10^{-4}$.

2. Solutions Along the Boundaries of the New Weyl-Like Chambers

When α and β are taken along the boundaries of the new chambers the solutions asymptotic to $-2/3z$ are nonoscillatory as $z \rightarrow -\infty$. Examples of this are shown in the center frames of figures 23 and 24. Now, if α or β are varied slightly such that the choice of parameters no longer falls on one of the boundaries, these solutions can have either an infinity of poles or oscillate as $z \rightarrow -\infty$. Examples of this are also shown in the left and right frames of figures 23 and 24.

3. When β is Positive

If $\beta > 0$, then figure 21 shows that all of the solutions asymptotic to $w_{+1}^- \sim -2/3z$ as $z \rightarrow +\infty$, $z \in \mathbb{R}$, have an infinity of poles on the negative real axis. These solutions also do not generally have an entire half-plane free of poles. Instead, numerical evidence points to a

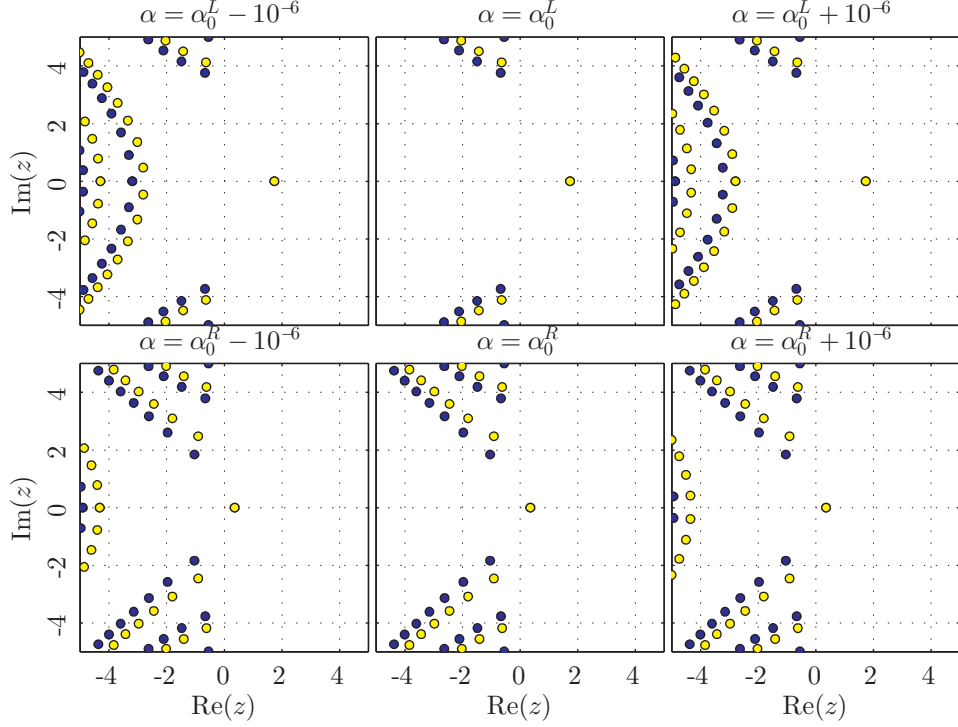


FIG. 24. Solutions (pole locations and residues) normal to the parabola $\beta = -2(\alpha + 2)^2$. All frames depict the solutions asymptotic to $-2/3z$ as $z \rightarrow +\infty$. The center frames occur directly along the parabolas where $\alpha = \alpha_0 = -1.25$ (top) and $\alpha = \alpha_0 = -2.75$ (bottom). The left and right frames in both the top and bottom then depict the solutions along the line normal to the parabola at $\alpha = \alpha_0$ at $\alpha_0 \pm 10^{-6}$.

value $z_0 \in \mathbb{R}$ (possibly positive or negative) such that for all z with $\text{Re}(z) > z_0$ the solution has no poles.

4. Other Solutions With a Pole Free Half-Plane

These solutions asymptotic to $-2/3z$ as $z \rightarrow +\infty$ are not the only solutions that have a half-plane pole free. There are, of course, the rational solutions. Likewise, there are solutions expressible in terms of parabolic cylinder or confluent hypergeometric functions that also feature a pole free half-plane. These solutions arise for $u_{\nu, \epsilon, d_1, d_2}^{[PC; k]}$, $k = 1, 2$, when either $d_1 = 0$ or $d_2 = 0$ with examples shown in figure 25. Generally, these other solutions with a pole free half-plane feature different asymptotics as $z \rightarrow +\infty$ than $-\frac{2}{3}z$.

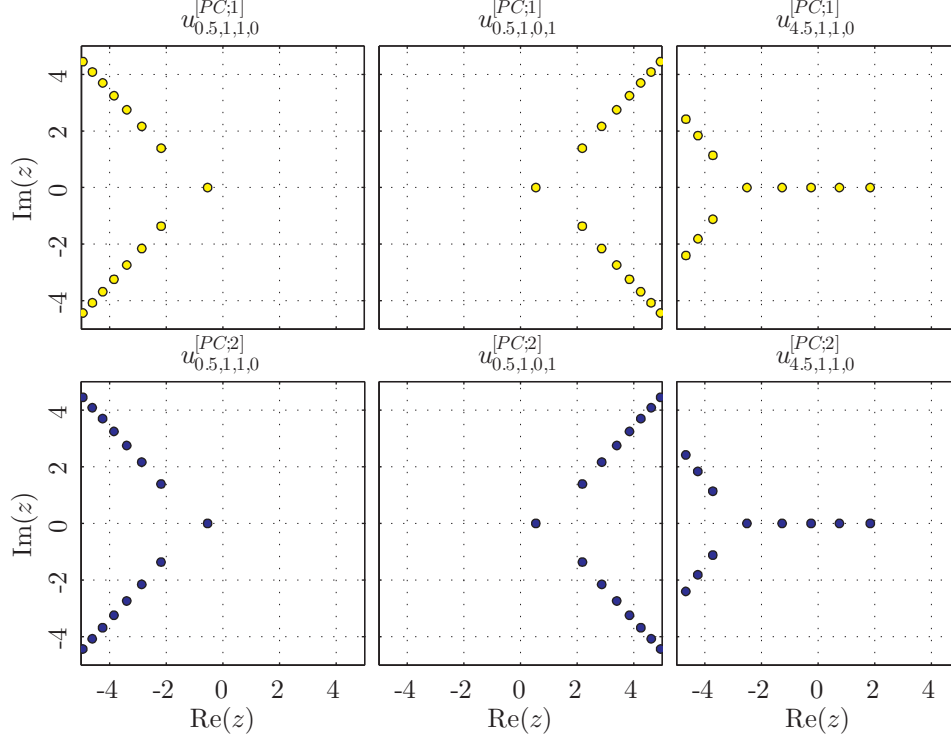


FIG. 25. Examples (pole locations and residues) of $u_{\nu,\epsilon,d_1,d_2}^{[PC;k]}$, $k = 1, 2$, for $d_1 = 0$ or $d_2 = 0$. These solutions feature a half plane that contains only a finite number of poles.

D. Solutions With Adjacent Pole Free Sectors

In Ref. 5 it is pointed out that there are solutions for P_{IV} when $\alpha = \beta = 0$ that are similar to the tronquée solutions of P_I . For both P_I and P_{IV} (with $\alpha = \beta = 0$) these solutions are characterized by at least two adjacent pole free sectors. In the case of P_{IV} these sectors are shown in figure 8. Also, when $\alpha = \beta = 0$, these solutions are characterized as appearing at the boundaries of shaded regions or along curves within the pole counting diagrams. From here on, the analogy with the tronquée solutions of P_I will be dropped and these solutions will be referred to only as having adjacent pole free sectors. The solutions asymptotic to $w_{+1}^- \sim -2/3z$ were considered separately in section VIC, but they would certainly fall into this category. Other solutions with adjacent pole free sectors are asymptotic to w_{-1}^- and w_{μ}^+ , $\mu = \pm 1$, as $z \rightarrow +\infty$ and $z \in \mathbb{R}$. In certain cases there are two or three such trends present simultaneously in a single solution, but the trends occur along different segments of the positive real axis. For instance, ICs generating solutions matching both w_{+1}^+ and w_{-1}^+ occur when $\beta = 0$. This is not surprising considering (14) and (15) and that these are simply the solutions asymptotic to (12). Several examples are available in⁵.

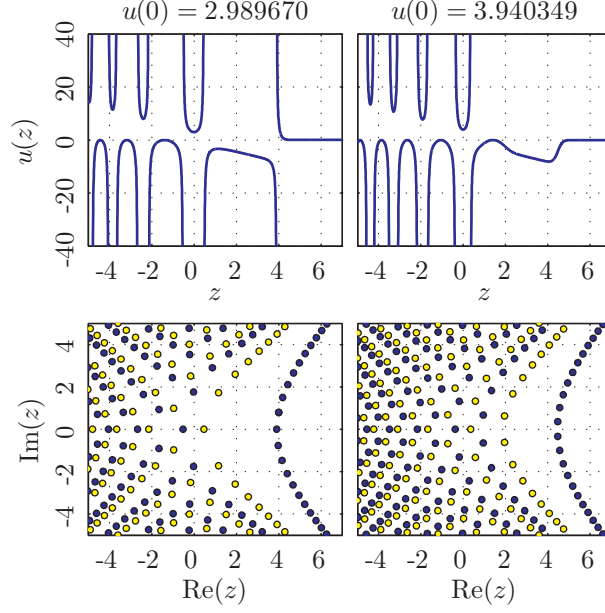


FIG. 26. Solution types (along the real axis (top) and pole locations and residues (bottom)) with adjacent pole free sectors for $\alpha = 1$ and $\beta = 0$. In all frames $u'(0) = 0$. The left and right frames both show that these solutions simultaneously match the roots (in different segments of the real axis) w_μ^+ , $\mu = \pm 1$, and w_{-1}^- .

In the following figures multiple frames will be shown depicting the different types of solutions with adjacent pole free sectors for each (α, β) pair discussed. In most cases, solutions where two or more behaviors appear in the same solution will be given in at least one frame. In every case, the solutions shown occur at the boundary of or along the curve located in the first shaded region extending from $u'(0) = 5$ to $u'(0) = -5$ in the right half plane (i.e. $u(0) > 0$) of the appropriate pole counting figure. These solutions are all given along the line $u'(0) = 0$.

First, figures 26 and 27 show two types of solutions where the asymptotic behaviors of w_μ^+ , $\mu = \pm 1$, and w_{-1}^- are simultaneously present (along different segments of the real axis) in a solution generated from a single IC. These are shown for $(\alpha = 1, \beta = 0)$ and $(\alpha = 0, \beta = -2)$.

On the other hand, solutions that match both the roots $w_{-1}^\pm$ (again, in different segments of the real axis) were observed along the boundary $\beta = -2(\alpha - 1)^2$. An example appears in figure 28 for the case $\alpha = 0.5$ and $\beta = -0.5$.

Finally, all other parameter choices with adjacent pole free sectors have distinct ICs that generate solutions asymptotic to each of the roots w_μ^+ , $\mu = \pm 1$, and w_{-1}^- as in the figure 29.

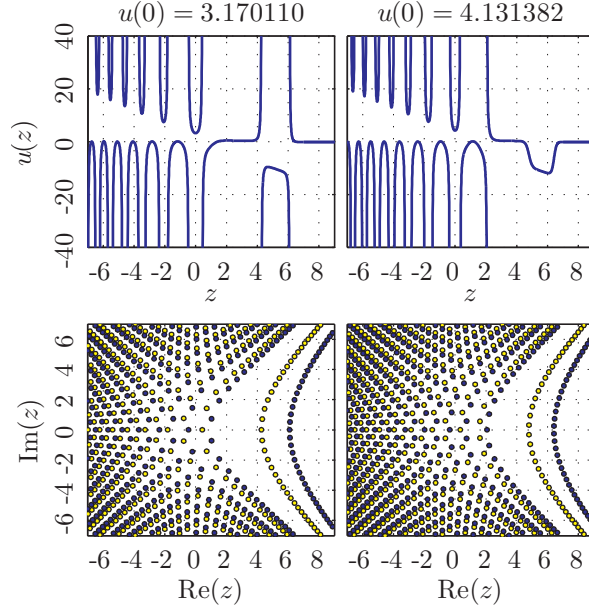


FIG. 27. Solution types (along the real axis (top) and pole locations and residues (bottom)) with adjacent pole free sectors for $\alpha = 0$ and $\beta = -2$. In all frames $u'(0) = 0$. The left and right frames both show that these solutions simultaneously match the roots w_μ^+ , $\mu = \pm 1$, and w_{-1}^- .

VII. CONCLUSIONS

This study of the fourth Painlevé equation started by numerically confirming various previous analytic and asymptotic results. A further exploration of the fundamental domain then identified solutions for general (α, β) -values with noteworthy characteristics, such as numerous families of solutions with adjacent pole-free sectors. Also, solutions with a nearly pole-free half plane were found.

Most of the observations in this study were obtained numerically, leaving analytical considerations of some of the illustrated solution types an open topic. Although the explorations extended outside of the fundamental domain in the (α, β) -plane, they considered only (α, β) -values with relatively small magnitude. Further studies could be performed to look at pairs with much larger magnitude. Another extension would be to also consider solutions that are complex-valued along the real axis.

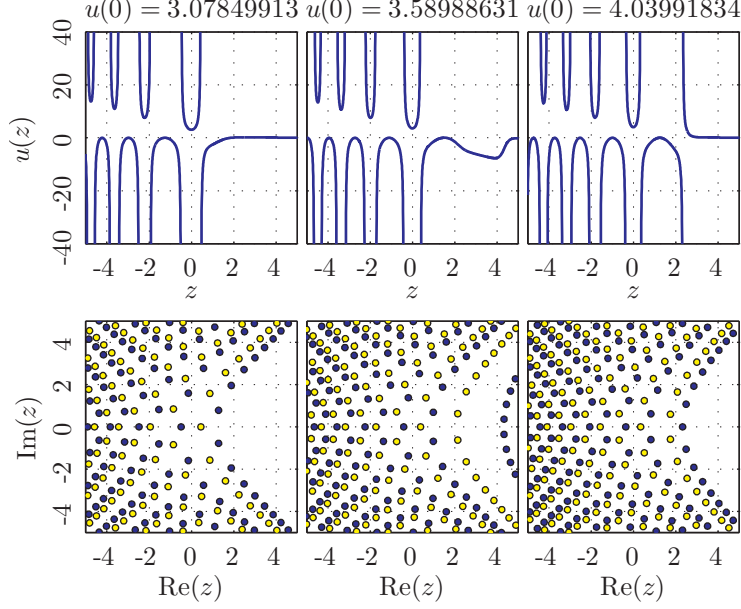


FIG. 28. Solution types (along the real axis (top) and pole locations and residues (bottom)) with adjacent pole free sectors for $\alpha = 0.5$ and $\beta = -0.5$. In all frames $u'(0) = 0$. The center frame shows that there are solutions simultaneously matching both the roots $w_{-1}^{\pm}$.

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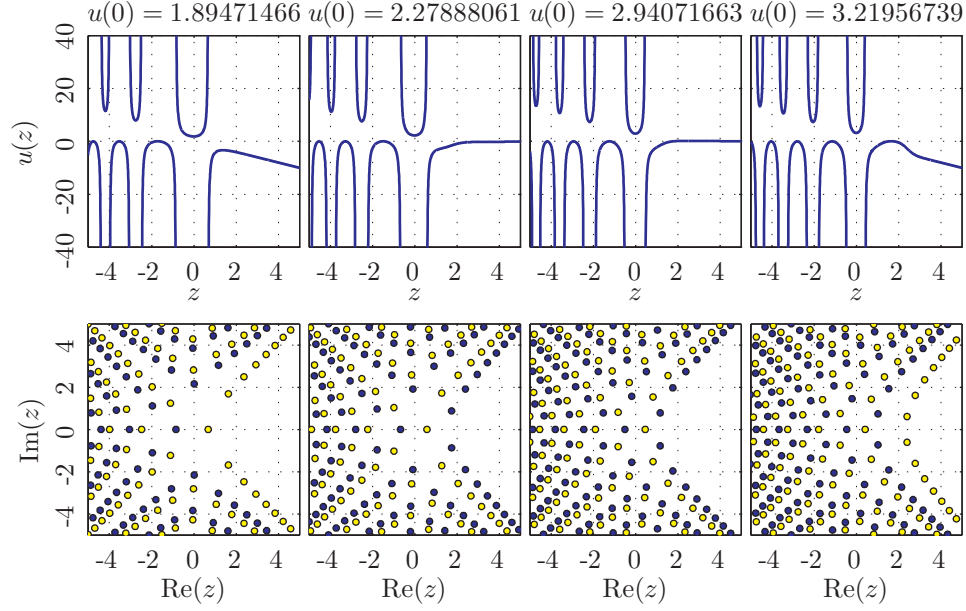


FIG. 29. Solution types (along the real axis (top) and pole locations and residues (bottom)) with adjacent pole free sectors for $\alpha = 0$ and $\beta = -0.5$. In all frames $u'(0) = 0$. In this case, all frames exhibit only one of the asymptotic behaviors $w_\mu^\pm$, $\mu = \pm 1$.

Plane (Walter de Gruyter, Berlin, 2002).

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