High-order accurate Nyström discretization of integral equations with weakly singular kernels on smooth curves in the plane

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Abstract: The note describes the construction of high-order accurate Nyström discretizations for the Boundary Integral Equations (BIEs) associated with the Laplace and Helmholtz equations in the plane. Only smooth boundaries are considered, but some of the techniques described can be extended to the piece-wise smooth case. The quadrature nodes are equispaced, or those associated with a composite Gaussian rule. In either case, a small number of elements in the coefficient matrix near the diagonal need to be modified to attain high-order accuracy. Three different approaches are considered: (1) Kolm-Rokhlin modification to composite Gaussian quadratures, (2) Kapur-Rokhlin modification to the trapezoidal rule and, (3) Alpert modification to the trapezoidal rule. The three approaches are described in detail, and several numerical experiments illustrating their relative advantages and drawbacks are presented.

1. Introduction

1.1. **Problem formulation.** This note describes quadrature rules for discretizing integral equations of the form

$$b\sigma(x) + \int_0^T k(x,x')\sigma(x')\,dx' = f(x), \qquad x \in [0,T]. \tag{1.1}$$

where $[0,T]$ is an interval, where f is a given smooth T-periodic function, and where k is a (doubly) T-periodic kernel function that is smooth away from the origin and has a logarithmic singularity as $x' \to x$. Specifically, we assume that

$$k(x,x') = \varphi(x,x')\log\left(\sin\frac{|x-x'|\pi}{T}\right) + \psi(x,x') \tag{1.2}$$

for some smooth T-periodic functions φ and ψ.

Integral equations such as (1.1) frequently arise in the study of Boundary Integral Equations (BIEs) such as

$$b\sigma(\boldsymbol{x}) + \int_\Gamma k(\boldsymbol{x},\boldsymbol{x}')\,\sigma(\boldsymbol{x}')\,dl(\boldsymbol{x}') = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma \tag{1.3}$$

where Γ is a smooth contour in the plane. The equation (1.1) is obtained from (1.3) by parameterizing the contour Γ, see Section 6.2.1 for details.

1.2. **Nyström discretization.** In the Nyström method for discretizing (1.1) one starts with a set of *nodes* $\{x_i\}_{i=1}^N$ such that

$$0 \le x_1 < x_2 < x_3 < \cdots < x_N < T$$

and constructs a linear system that relates the given data vector $\boldsymbol{f} = \{f_i\}_{i=1}^N$ where $f_i = f(x_i)$ to an unknown solution vector $\boldsymbol{\sigma} = \{\sigma_i\}_{i=1}^N$ where $\sigma_i \approx \sigma(x_i)$. Informally speaking, the idea is to use the nodes $\{x_i\}_{i=1}^N$ as *collocation points* where (1.1) is enforced:

$$b\sigma(x_i) + \int_0^T k(x_i,x')\sigma(x')\,dx' = f(x_i), \qquad i = 1,\,2,\,3,\,\ldots,\,N. \tag{1.4}$$

Then coefficients $\{a_{i,j}\}_{i,j=1}^N$ are constructed such that for smooth σ

$$\int_0^T k(x_i,x')\sigma(x')\,dx' \approx \sum_{j=1}^N a_{i,j}\,\sigma(x_j). \tag{1.5}$$

Combining (1.4) and (1.5) we obtain a square linear system that relates $\boldsymbol{\sigma}$ to $\boldsymbol{f}$:

$$b\sigma_i + \sum_{j=1}^{N} a_{i,j}\,\sigma_j = f_i, \qquad i = 1,\, 2,\, 3,\, \ldots,\, N. \tag{1.6}$$

We write (1.6) in matrix form as

$$b\boldsymbol{\sigma} + \mathsf{A}\boldsymbol{\sigma} = \boldsymbol{f}. \tag{1.7}$$

1.3. **Quadrature rules for Nyström discretization.** In order to minimize the discretization error in the Nyström scheme, we seek nodes $\{x_i\}_{i=1}^N$ and coefficients $\{a_{i,j}\}_{i,j=1}^N$ such that (1.5) holds to high accuracy. If the kernel function k is smooth (this is not typical), then this task is easy: Simply pick a quadrature rule for the interval $[0,\, T]$ with nodes $\{x_i\}_{i=1}^N$ and weights $\{w_i\}_{i=1}^N$, and set

$$a_{ij} = k(x_i, x_j)w_j. \tag{1.8}$$

The purpose of this note is to describe and compare a few different techniques for constructing nodes $\{x_i\}$ and coefficients $\{a_{i,j}\}$ such that (1.5) holds to high accuracy in the more typical case where k has a singularity at the diagonal such as (1.2). It is very desirable to have the coefficients $a_{i,j}$ be given by a simple formula such as (1.8), and the technique described in Section 3 comes very close to achieving this goal, cf. (3.5). Sections 4 and 5 describe two techniques for which (1.8) holds except for a small number of entries located in a narrow band around the diagonal.

The principal reason we seek techniques that preserve (1.8) for as many entries as possible is that there exist $O(N)$ methods for evaluating highly accurate approximations to N^2-term sums such as

$$\sum_{j=1}^{N} k(x_i, x_j)\, q_j, \qquad i = 1,\, 2,\, 3,\, \ldots,\, N,$$

for many of the standard kernels arising in potential theory. The most well-known is probably the *Fast Multipole Method* [8] by Rokhlin and Greengard, but others exist [3, 7]. When (1.8) holds, such methods permit $O(N)$ application of the matrix A via the formula

$$[\mathsf{A}\boldsymbol{\sigma}](i) = \sum_{j=1}^{N} k(x_i, x_j)\,\big(w_j\sigma_j\big), \qquad i = 1,\, 2,\, 3,\, \ldots,\, N.$$

This is essential since a linear system such as (1.7) is often solved using iterative methods (such as, e.g., GMRES) when N is large. Note that using a different formula for coefficients $a_{i,j}$ located close to the diagonal is unproblematic since this involves a modification of only $O(N)$ terms in the sum.

Remark 1.1. A reason high-order Nyström schemes can be constructed is that even though the integrand in (1.3) is non-smooth, it turns out that in many situations of interest (e.g. in potential theory) the solution σ is smooth. The quadratures are based on approximating σ by interpolating functions (polynomials or piecewise polynomials for the methods under consideration here, but other families of functions could be used as well), and then applying the integral operator to the interpolant.

1.4. **Related work.** The methods described in this paper rely on earlier work [1, 10, 11] describing high-order quadrature rules for integrands with weakly singular kernels. It appears likely that these rules were designed in part to facilitate Nyström discretization of BIEs, but the original papers leave the details out. Some later papers reference the use (e.g. [4, 12]) of high order quadratures but provide few details. In particular, there appear to have been no formal comparison between the accuracy of different approaches. An informal comparison was recently done by Alexander Barnett of Dartmouth College, and the present note can be viewed as our continuation of that investigation.

1.5. **Outline of paper.** Section 2 introduces necessary mathematical and numerical machinery. Sections 3, 4 and 5 describe three numerical quadrature rules: Kapur-Rokhlin quadrature, modified Gaussian quadrature and Alpert quadrature applicable to kernels with integrable singularities. These sections describe in detail the Nyström schemes associated with the three quadratures. Section 6 presents the results of numerical experiments conducted to compare the performance of each quadrature rule.

2. mathematical preliminaries

In this section, we summarize several results from numerical analysis to be used throughout this note. Detailed references are given in the text.

2.1. **Gaussian Quadratures.** A quadrature rule on the interval $[-1,\, 1]$ is an expression of the form

$$\sum_{j=1}^{N} w_j f(x_j), \tag{2.1}$$

where the points $\{x_j\}_{j=1}^N \in [-1,\, 1]$ and the coefficients $\{w_j\}_{j=1}^N \in \mathbb{R}$ are referred to as the quadrature nodes and weights. The quadrature rule (2.1) serves as an approximation to integrals of the form

$$\int_{-1}^{1} w(x) \cdot f(x) dx, \tag{2.2}$$

where $f : [-1,\, 1] \to \mathbb{R}$ is a sufficiently smooth function and $w : [-1,\, 1] \to \mathbb{R}$ is some fixed weight function.

It is common to require a quadrature rule to integrate all polynomials up to some certain degree exactly. For a rule with N points, the best performance possible in this regard is to integrate all polynomials up to degree $2N - 1$. This optimal performance is attained by so called "Gaussian quadratures". The following theorem is well known and can be found in most elementary textbooks on numerical analysis (see, for example, [2])

Theorem 2.1 (Gaussian Quadrature)**.** *Suppose that $w(x) \equiv 1$ for all $x \in [-1,\, 1]$. Then there exists a unique quadrature rule (2.1) that exactly integrates all polynomials of degree at most $2N-1$. Furthermore, the nodes $\{x_j\}_{j=1}^N$ are the zeros of the N^{th} Legendre polynomials $P_N(x)$, and the weights $\{w_j\}_{j=1}^N$ are all positive and given by the formula*

$$w_j = \frac{-2}{(N+1)P_N'(x_j)P_{N+1}(x_j)}, \qquad j = 1, \ldots, N.$$

2.2. **Lagrange interpolation.** For a given set of distinct nodes $\{x_j\}_{j=1}^N$ and values $\{y_j\}_{j=1}^N$, the Lagrange interpolation polynomial $L(x)$ is a polynomial of degree no greater than $N-1$ that passes through the N points $\{(x_j,\, y_j)\}_{j=1}^N$, and is given by

$$L(x) = \sum_{j=1}^{N} y_j\, L_j(x),$$

where

$$L_j(x) = \prod_{\substack{i=1 \\ i \neq j}}^{N} \left(\frac{x - x_i}{x_j - x_i} \right). \tag{2.3}$$

In particular, let $\{x_j\}_{j=1}^N$ denote the nodes of an N-point Gaussian quadrature rule on interval $[0,\, b]$. If the values of a polynomial f of degree at most $N-1$ are specified at these nodes, the entire

polynomial f can be recovered via the formula

$$f(s) = \sum_{j=1}^{N} L_j(s)\, f(x_j),$$

where the functions L_j are the Lagrange basis polynomials (2.3).

If f is a smooth function with N continuous derivatives, then the Lagrange interpolant provides an approximation to f satisfying

$$\left| f(s) - \sum_{j=1}^{N} L_j(s)\, f(x_j) \right| \le C\, b^N,$$

where

$$C = \left(\sup_{s\in[0,\,b]} |f^{(N)}(s)| \right) / N!.$$

3. Nyström discretization using the Kapur-Rokhlin quadrature rule

3.1. **The Kapur-Rokhlin quadrature rule.** In this section, we seek to approximate the integral over $[0,\, b]$ of functions $g(x) : (0,\, b] \to R$ that take the form

$$g(x) = \varphi(x)s(x) + \psi(x), \tag{3.1}$$

where $\varphi(x), \psi(x) \in C^\infty[0,\, b]$ and $s(x) \in C(0,\, b]$ is an integrable function with a singularity at zero. The standard trapezoidal rule designed for smooth functions would lose almost all accuracy at the singularity. However, this problem can be remedied by executing the Kapur-Rokhlin quadrature rule developed by Sharad Kapur and Vladimir Rokhlin. Briefly speaking, this procedure described in [10] starts with a standard trapezoidal rule and modifies the weights near the sigular points to achieve high order convergence.

Specifically, to integrate functions g over $[0,\, b]$ of form (3.1) where $\varphi, \psi \in C^\infty[-mh,\, b]$, an m^{th}-order Kapur-Rokhlin rule $T^N_{\gamma^m}$ is given by the formula

$$T^N_{\gamma^m}(g) = T_N(g) + h \sum_{\substack{\ell=-m \\ \ell\neq 0}}^{m} \gamma_\ell\, g(\ell h), \tag{3.2}$$

where

$$h = \frac{b}{N-1},$$

and T_N is the N-point trapezoidal rule skipping the singular point, given by the formula

$$T_N(g) = h\left[g(h) + \cdots + g(b-h) + \frac{1}{2} g(b) \right], \tag{3.3}$$

and $\{\gamma_\ell\}^m_{\ell=-m,\ell\neq 0}$ are the correction weights around the end points (singular points).

For any $\varphi, \psi \in C^\infty[-mh,\, b]$

$$\left| \int_0^b g(x)\, dx - T^N_{\gamma^m}(g) \right| = O(h^m). \tag{3.4}$$

Remark 3.1. Compared with some other endpoint corrected schemes whose correction weights grow rapidly with orders rendering the schemes useless, the growth of correction weights of Kapur-Rokhlin rule is suppressed. This enables the quadrature to integrate smooth functions to arbitrarily high order and functions of the form (3.1) up to the order of 12. The principal drawback of Kapur-Rokhlin rule is that it requires a small amount of function values *outside* the integration interval, which sometimes makes the method inapplicable. However, for the problems we are dealing with

such as the kernel function defined in (1.2), the values of the integrand are always available on both sides of the singularities. In fact, in this case the functions are periodic since they live on closed curves.

3.2. **A Nyström scheme.** Using the tools reviewed in Section 3.1, we are now in position to construct numbers $a_{i,j}$ such that (1.5) holds. The starting point is to enforce the set of nodes $\{x_i\}_{i=1}^N$ to be equispaced nodes $\{(i-1)h\}_{i=1}^N \subset [0,\, T]$ where $h = T/(N-1)$ and to apply the trapezoidal rule for the interval $[0,\, T]$ with nodes $\{x_j\}_{j=1}^N = \{(j-1)h\}_{j=1}^N$ such that

$$\int_0^T k(x_i, x')\sigma(x')\,dx' \approx h\sum_{j=1}^N k(x_i, x_j)\sigma(x_j), \quad i = 1, 2, \ldots, N.$$

The kernel function $k(x_i, x_j)$ associated with BIEs typically has a singularity when $x_j = x_i$, for example, a log-singularity associated with Laplace equation. To overcome the poor approximation around the singularities, an m^{th}-order Kapur-Rokhlin correction is executed. To this end, we first introduce a distance function $\ell(i, j)$ between two nodes x_i and x_j defined by

$$\ell(i, j) \equiv j - i \pmod N,$$

and two nodes x_i and x_j are said to be "close" if $|\ell(i, j)| \le m$. Moreover, we call x_i and x_j "well-separated" if they are not "close". On the other hand, given $\ell \in \{-m, -m+1, \ldots, m\}$, we have a unique index $j \in \{1, 2, \ldots, N\}$ such that $\ell(i, j) = \ell$.

Now based on the standard trapezoidal rule skipping node x_i, we add correction weights to those nodes that are "close" to x_i and leave the weights of those nodes that are "well-separated" from x_i unchanged. Remember the kernel function in (1.5) is T-periodic such that the function values are always available on both sides of x_i by continuing the integrand periodically. In this way, we have the integral in (1.5) evaluated as

$$\begin{aligned}
&\int_0^T k(x_i, x')\sigma(x')\,dx' \\
&\approx h\sum_{\substack{j=1\\ j\neq i}}^N k(x_i, x_j)\sigma(x_j) + h\sum_{\ell=1}^m \gamma_\ell\, k(x_i, x_i+\ell h)\,\sigma(x_i+\ell h) + h\sum_{\ell=1}^m \gamma_{-\ell}\, k(x_i, x_i+T-\ell h)\,\sigma(x_i+T-\ell h) \\
&= h\sum_{\substack{j=1\\ j\neq i}}^N k(x_i, x_j)\sigma(x_j) + h\sum_{\substack{\ell(i,j)=-m\\ \ell(i,j)\neq 0}}^m \gamma_{\ell(i,j)}\, k(x_i, x_j)\,\sigma(x_j).
\end{aligned}$$

In particular, the entries of coefficient matrix A are given by

$$a_{i,j} = \begin{cases} 0 & \text{if } i = j, \\ h\,k(x_i, x_j) & \text{if } x_i \text{ and } x_j \text{ are "well-separated"}, \\ h\,(1+\gamma_{\ell(i,j)})\,k(x_i, x_j) & \text{if } x_i \text{ and } x_j \text{ are "close", and } i \neq j, \end{cases} \tag{3.5}$$

in which $x_i = (i-1)h$ and $x_j = (j-1)h$ for $i, j = 1, 2, \ldots, N$ and $h = T/(N-1)$.

4. Nyström discretization using modified Gaussian quadrature

4.1. **Modified Gaussian quadrature.** Suppose that given an interval $[0,\, b]$ and a point $t \in [-b,\, 2b]$, we seek to evaluate integrals of functions g that take the form

$$\int_0^b g(s)\,ds = \int_0^b \big(\varphi(s)\,k(t,\, s) + \psi(s)\big)\,ds, \tag{4.1}$$

where φ and ψ are smooth functions over $[0,\, b]$ and $k(t,\, s)$ has a singularity as $s \to t$. Standard Gaussian quadrature (see Theorem 2.1) would be inaccurate if applied to evaluate (4.1). Instead, for t being a point located in $[0,\, b]$, we seek an m-node modified Gaussian quadrature rule $(\{v_k\}_{k=1}^m, \{y_k\}_{k=1}^m \subset [0,\, b])$ that evaluates the integral (4.1) to very high accuracy. In particular, we use a quadrature formula of the form

$$\int_0^b \big(\varphi(s)\, k(t,\, s) + \psi(s)\big)\, ds \approx \sum_{k=1}^{m} v_k \big(\varphi(y_k)\, k(t,\, y_k) + \psi(y_k)\big) \tag{4.2}$$

which holds when φ and ψ are polynomials of degree n. Another special case is when t locating at the neighborhood of s, for example $t \in [-b,\, 0] \cup [b,\, 2b]$, the kernel function is smooth but has a sharp peak rendering standard Guassian quadrature inaccurate. In this case, we can approximate the integral (4.1) with another set of modified Gaussian quadrature weights and nodes $(\{\hat{v}_k\}_{k=1}^{m'}, \{\hat{y}_k\}_{k=1}^{m'} \subset [0,\, b])$ giving a quadrature formula totally analogous to (4.2). Notice that in general $m, m' > n$. Techniques for constructing such modified quadratures are available in [11].

4.2. **A Nyström scheme.** In order to obtain a formula like (1.5) that can approximate the integral

$$\int_0^T k(x_i, x')\sigma(x')\, dx' \tag{4.3}$$

to high accuracy, we proceed as follows. First, we partition the domain of integration as

$$[0,\, T] = \bigcup_{p=1}^{N_P} \Omega_p,$$

where the Ω_p's are non-overlapping subintervals called panels. For simplicity, we for now assume that the panels are equisized so that $\Omega_p = \left[\frac{T(p-1)}{N_P},\, \frac{Tp}{N_P}\right]$. On each panel, we place the nodes of an n-point Gaussian quadrature to obtain a total of $N = n\, N_P$ nodes. Let $\{x_i\}_{i=1}^N$ and $\{w_i\}_{i=1}^N$ denote the nodes and weights of the resulting composite Gaussian rule.

Next, fix a node x_i and let Ω_p be the panel holding x_i. We decompose the integral we seek to approximate, cf. (4.3), as

$$\int_0^T k(x_i, x')\, \sigma(x')\, dx' = \sum_{q=1}^{N_P} \int_{\Omega_q} k(x_i, x')\, \sigma(x')\, dx'. \tag{4.4}$$

Our task is now to construct an approximation for each panel-wise integral

$$\int_{\Omega_q} k(x_i, x')\sigma(x')\, dx', \tag{4.5}$$

expressed in terms of the values of σ at the nodes in Ω_q. There are three distinct cases:

Case 1: x_i belongs to the source panel Ω_q: The integrand is now singular in the domain of integration, but we can exploit that σ is still smooth and can be approximated via polynomial interpolation:

$$\sigma(x') \approx \sum_{j\,:\; x_j \in \Omega_q} L_j(x')\, \sigma(x_j). \tag{4.6}$$

Inserting (4.6) into the integral in (4.5) we find

$$\int_{\Omega_q} k(x_i, x')\sigma(x')\, dx' \approx \sum_{j\,:\; x_j \in \Omega_q} \left(\int_{\Omega_q} k(x_i, x') L_j(x')\, dx' \right) \sigma(x_j) dx'.$$

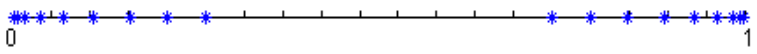


FIGURE 1. Correction nodes $\{\chi_p h\}_{p=1}^m$ and $\{1-\chi_p h\}_{p=1}^m$ for $m = 10$ and $a = 6$ at endpoints 0 and 1, denoted by stars.

Approximating the singular integrals using the rule (4.2) we find

$$\int_{\Omega_q} k(x_i, x')\sigma(x')\,dx' \approx \sum_{j\,:\,x_j\in\Omega_q} \left(\sum_{k=1}^{m} v_{i,k}\,k(x_i, y_{i,k})\,L_j(y_{i,k})\right)\sigma(x_j). \tag{4.7}$$

Case 2: x_i belongs to a panel Ω_p directly adjacent to panel Ω_q: In this case, the kernel k is smooth, but very sharply peaked. We therefore proceed as in Case 1: Replace σ by its polynomial interpolant and then integrate using the modified quadratures described in Section 4.1. The end result is a formula entirely analogous to (4.7).

Case 3: x_i is well-separated from the source panel Ω_q: In this case, both the kernel k and the potential σ are smooth, so the original Gaussian rule will be accurate,

$$\int_{\Omega_q} k(x_i, x')\sigma(x')\,dx' \approx \sum_{j\,:\,x_j\in\Omega_q} w_j k(x_i, x_j)\sigma(x_j). \tag{4.8}$$

Combining (4.8) and (4.7), we now find $a_{i,j}$ is given by

$$a_{i,j} = \begin{cases} \sum_{k=1}^{m} v_{i,k}\,k(x_i, y_{i,k})\,L_j(y_{i,k}) & \text{if } x_i \text{ and } x_j \text{ belong to identical or adjacent panels,} \\ k(x_i, x_j)\,w_j & \text{if } x_i \text{ and } x_j \text{ belong to well-separated panels.} \end{cases} \tag{4.9}$$

5. Nyström discretization using the Alpert quadrature rule

5.1. **The Alpert quadrature rules.** Alpert quadrature is another efficient numerical method to approximate the integral over $[0,\, b]$ of functions that take the form

$$g(x) = \varphi(x)s(x) + \psi(x), \tag{5.1}$$

where $\varphi(x), \psi(x) \in C^\infty[0,\, b]$ and $s(x) \in C(0,\, b)$ is an integrable function with singularities at both sides of the interval, say 0 and b. It is constructed based on the trapezoidal rule, in which a small number of nodes and weights at the ends of the integration interval are replaced.

Specifically, for function g in (5.1), we will be denoting by $S_{\chi^m}^N(g)$ the m^{th}-order Alpert quadrature rule, defined by the formula

$$S_{\chi^m}^N(g) = h\sum_{p=1}^{m} w_p\,g(\chi_p\,h) + h\sum_{j=0}^{N-2a} g(ah + jh) + h\sum_{p=1}^{m} w_p\,g(b - \chi_p\,h). \tag{5.2}$$

In (5.2), there are $N - 2a + 1$ "internal" equally spaced nodes with spacing $h = b/(N-1)$ and m correction nodes replacing the original $a - 1$ equispaced nodes at each end. The endpoint correction numbers $\chi_1, \ldots \chi_m$ and weights $w_1, \ldots w_m$ are chosen so that $S_{\chi^m}^n$ is a quadrature rule with error of order $O(h^{m+1}\log h)$. For each choice of m, the integer a is chosen by experiments to be the smallest integer leading to positive correction nodes and weights. Details on how to construct these numbers can be found in [1]. Figure 5.1 depicts the locations of 10 correction nodes $\{\chi_p h\}_{p=1}^{10}$ and $\{1-\chi_p h\}_{p=1}^{10}$ superimposed on 5 original equispaced nodes ($m = 10$, $a = 6$) when the singularities occur at the two endpoints of interval $[0,\, 1]$, as an example.

5.2. **A Nyström scheme.** To implement the m^{th}-order Alpert scheme with parameter a, we start with continuing the integral in (1.5) periodically with period T around x_i such that

$$\int_0^T k(x_i,x')\sigma(x')\,dx' = \int_{x_i}^{x_i+T} k(x_i,x')\sigma(x')\,dx', \quad i=1,2,\ldots,N,$$

where $\{x_i\}_{i=1}^N = \{(i-1)h\}_{i=1}^N$ with spacing $h = T/(N-1)$. The kernel function $k(x_i,x')$ is singular when $x' = x_i$ and $x' = x_i + T$ rendering the standard trapezoidal rule inaccurate. But this can be remedied by modifying the nodes and weights at the two end points of the shifted interval $[x_i, x_i+T]$. To this end, we define the distance between two nodes x_i and x_j via

$$\ell(i,j) \equiv j - i \pmod N.$$

Then we replace the nodes x_j that are "close" to x_i in the sense that $|\ell(i,j)| < a$ by m correction nodes $\{x_i + \chi_p h\}_{p=1}^m$ associated with weights $\{w_p\}_{p=1}^m$. Modifying the nodes and weights that are "close" to $x_i + T$ can be done analogously with correction nodes $\{x_i + T - \chi_p h\}_{p=1}^m$ and weights $\{w_p\}_{p=1}^m$. In this way, we can compute the integral in (1.5) as following

$$\begin{aligned}\int_0^T k(x_i,x')\sigma(x')\,dx' \approx\ & h\sum_{p=0}^{N-2a} k(x_i, x_i+ah+ph)\,\sigma(x_i+ah+ph)\\ &+ h\sum_{p=1}^{m} w_p k(x_i, x_i+\chi_p h)\,\sigma(x_i+\chi_p h)\\ &+ h\sum_{p=1}^{m} w_p k(x_i, x_i+T-\chi_p h)\,\sigma(x_i+T-\chi_p h).\end{aligned} \tag{5.3}$$

Generally, the endpoint correction numbers $\{\chi_p\}_{p=1}^m$ are not integers. In order to have a quadrature evaluated at equispaced nodes, we interpolate the function values of σ at nodes $\{x_i+\chi_p h\}_{p=1}^m$ and $\{x_i+T-\chi_p h\}$ with $(m+3)+1$ equispaced nodes around x_i and x_i+T. Let $\{L_q^{(x_i)}\}_{q=0}^{m+3}$ denote the local Lagrange interpolation functions of order $m+3$ defined on the neighborhood of node x_i. Then

$$\sigma(x) \approx \sum_{q=0}^{m+3} L_q^{(x_i)}(x)\sigma(x_i+qh),$$

where $x \in [x_i, x_i+(m+3)h]$ and

$$L_q^{(x_i)}(x) = \prod_{\substack{r=0\\ r\neq q}}^{(m+3)} \frac{x-(x_i+rh)}{(x_i+qh)-(x_i+rh)}, \qquad q=0,\ldots,m+3.$$

Therefore

$$\sigma(x_i+\chi_p h) \approx \sum_{q=0}^{m+3} L_q^{(x_i)}(x_i+\chi_p h)\sigma(x_i+qh) \tag{5.4}$$

The local Lagrange interpolation functions $L_q^{(x_i+T)}$ can be defined analogously on the neighborhood of x_i+T and

$$\sigma(x_i+T-\chi_p h) \approx \sum_{q=0}^{m+3} L_q^{(x_i+T)}(x_i+T-\chi_p h)\sigma(x_i+T-qh) \tag{5.5}$$

Inserting (5.4) and (5.5) into (5.3), we find that

(5.6)

$$\begin{aligned}\int_0^T k(x_i,x')\sigma(x')\,dx' \approx\, & h\sum_{p=0}^{N-2a} k(x_i,x_i+ah+ph)\,\sigma(x_i+ah+ph)\\ & + h\sum_{p=1}^{m} w_p k(x_i,x_i+\chi_p h)\sum_{q=0}^{m+3} L_q^{(x_i)}(x_i+\chi_p h)\sigma(x_i+qh)\\ & + h\sum_{p=1}^{m} w_p k(x_i,x_i+T-\chi_p h)\sum_{q=0}^{m+3} L_q^{(x_i+T)}(x_i+T-\chi_p h)\sigma(x_i+T-qh)\\ =\, & h\sum_{p=0}^{N-2a} k(x_i,x_i+ah+ph)\,\sigma(x_i+ah+ph)\\ & + h\sum_{q=0}^{m+3}\left(\sum_{p=1}^{m} w_p\,k(x_i,x_i+\chi_p h)\,L_q^{(x_i)}(x_i+\chi_p h)\right)\sigma(x_i+qh)\\ & + h\sum_{q=0}^{m+3}\left(\sum_{p=1}^{m} w_p\,k(x_i,x_i+T-\chi_p h)\,L_q^{(x_i+T)}(x_i+T-\chi_p h)\right)\sigma(x_i+T-qh).\end{aligned}$$

Notice that typically we have $m > a$. We now find the coefficient matrix A has entries

$$a_{i,j} = b_{i,j} + c_{i,j}, \tag{5.7}$$

where

$$b_{i,j} = \begin{cases} 0 & \text{if } |\ell(i,j)| < a,\\ h\,k(x_i,x_j) & \text{if } |\ell(i,j)| \geq a,\end{cases} \tag{5.8}$$

and

(5.9)

$$c_{i,j} = \begin{cases} 0 & \text{if } |\ell(i,j)| > m+3,\\ h\sum_{p=1}^{m} w_p\Big(k(x_i,x_i+\chi_p h)\,L_{\ell(i,j)}^{(x_i)}(x_i+\chi_p h) + k(x_i,x_i+T-\chi_p h)\,L_{\ell(i,j)}^{(x_i+T)}(x_i+T-\chi_p h)\Big) & \\ & \text{if } |\ell(i,j)| \leq m+3.\end{cases}$$

6. Numerical experiments

This section describes several numerical experiments that illustrate the performance of the quadratures described in Sections 3 - 5. They were carried out on a Macbook Pro with 2.4GHz Intel Core 2 Duo and 4GB of RAM. Codes for the experiments were written in Matlab. In all examples below, the errors reported are relative errors measured in the l^∞-norm, $||u_\epsilon - u||_\infty/||u||_\infty$, where u is the reference solution and u_ϵ is the numerical solution. In each experiment we compare the performance of different quadratures. Specifically, we compare the modified Gaussian quadrature with 10 points per panel to the 2^{nd}-order, 6^{th}-order and 10^{th}-order Kapur-Rokhlin and Alpert quadratures. All the quadrature nodes and weights used are listed in appendices A and B.

6.1. **A 1D example.**

In the first experiment, we solve the one-dimensional integral equation

$$u(x) + \int_{-\pi}^{\pi} k(x,x')u(x')dx' = f(x), \qquad x \in I = [-\pi,\,\pi] \tag{6.1}$$

associated with kernel function having a singularity at the diagonal

$$k(x,x') = \log\left|\sin\frac{x-x'}{2}\right| = \frac{1}{2}\log\left(\frac{1-\cos(x-x')}{2}\right). \tag{6.2}$$

Choose f a periodic function on $[-\pi,\,\pi]$, for example $f(x) = \sin(3x)\,e^{\cos(5x)}$. Since we are aware of no way to explicitly compute the exact solution of (6.1), we compute the difference between subsequent solutions. (Note that since the composite Gaussian rule does not have "nested nodes" interpolation was used to evaluate the solution u at some test points.) In Figure 6.1, the errors measured in l^∞-norm are presented for $N = 20, 40, \ldots, 2\,560$. Notice that in the Nyström discretization associated with modified Gaussian quadrature in all the numerical examples, we first split the interval of integration into several panels and on each panel a 10-point standard Gaussian quadrature was used. Meanwhile, the 20-point and 24-point modified Gaussian quadratures were used when x' is on the same panel of x and on the adjacent panels of x, respectively.

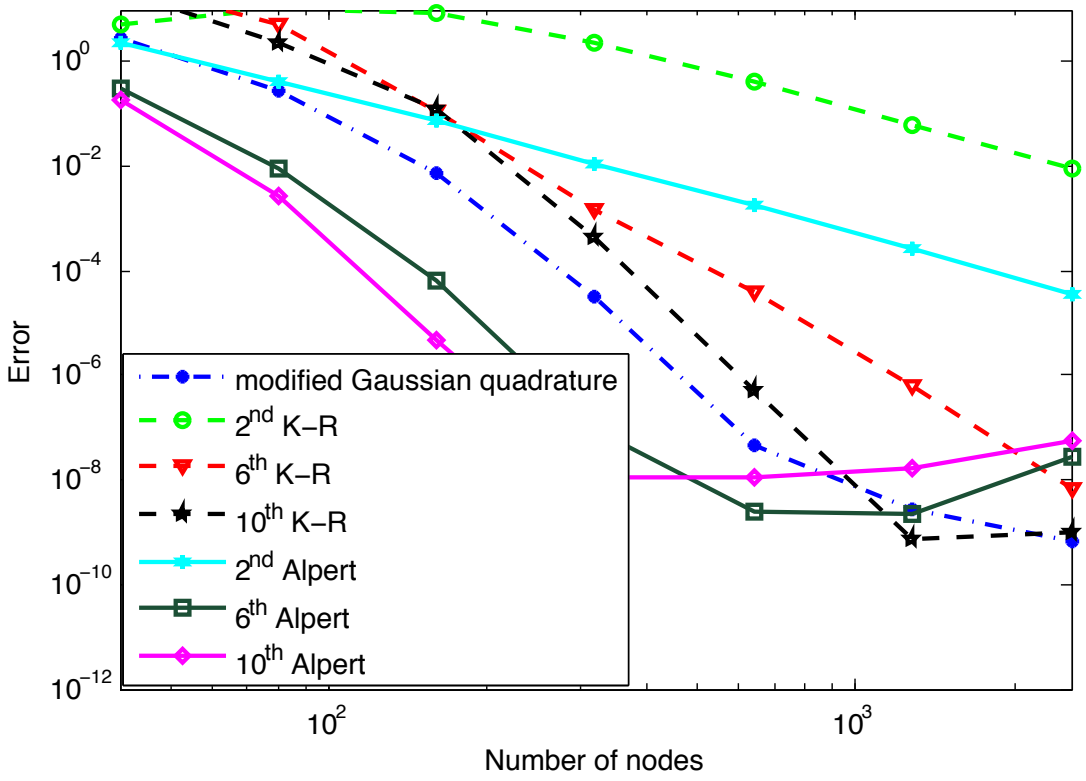


FIGURE 2. Error results for solving the integral equation (6.1) in Section 6.1.

6.2. **A BIE for Laplace's equation with Dirichlet boundary data.** In this section, we apply three quadrature rules of different orders to interior Dirichlet problem for Laplace's equation

$$-\Delta u = 0, \qquad \text{in } \Omega, \tag{6.3}$$

$$u = f, \qquad \text{on } \Gamma \tag{6.4}$$

associated with the combination of single and double layer kernels

$$u(\boldsymbol{x}) = \int_\Gamma D(\boldsymbol{x}, \boldsymbol{x}')\, \sigma(\boldsymbol{x}')\, dl(\boldsymbol{x}') + \int_\Gamma S(\boldsymbol{x}, \boldsymbol{x}')\, \sigma(\boldsymbol{x}')\, dl(\boldsymbol{x}')$$

where

$$S(\boldsymbol{x}, \boldsymbol{x}') = -\frac{1}{2\pi} \log |\boldsymbol{x} - \boldsymbol{x}'|,$$
$$D(\boldsymbol{x}, \boldsymbol{x}') = -\frac{1}{2\pi} \frac{\partial \log |\boldsymbol{x} - \boldsymbol{x}'|}{\partial \boldsymbol{n}(\boldsymbol{x}')},$$

and $\boldsymbol{n}(\boldsymbol{x}')$ is the outward-pointing unit normal vector to Γ at point $\boldsymbol{x}'$.

The corresponding boundary integral equation is given by

$$-\frac{1}{2}\sigma(\boldsymbol{x}) + \int_\Gamma k(\boldsymbol{x}, \boldsymbol{x}')\sigma(\boldsymbol{x}')\, dl(\boldsymbol{x}') = f(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Gamma, \tag{6.5}$$

where $k(\boldsymbol{x}, \boldsymbol{x}') = S(\boldsymbol{x}, \boldsymbol{x}') + D(\boldsymbol{x}, \boldsymbol{x}')$.

6.2.1. *Parametrization of the curve.* Let Γ be parameterized by a vector-valued smooth function $\boldsymbol{\tau} : [0, T] \to \mathbb{R}^2$. The parametrization recasts (6.5) as an integral equation on the interval $[0, T]$:

$$-\frac{1}{2}\sigma(\boldsymbol{\tau}(t)) + \int_0^T k(\boldsymbol{\tau}(t), \boldsymbol{\tau}(s))\, \sigma(\boldsymbol{\tau}(s))\, |d\boldsymbol{\tau}/ds|\, ds = f(\boldsymbol{\tau}(t)), \qquad t \in [0, T]. \tag{6.6}$$

To keep our formula uncluttered, we rewrite the kernel as

$$m(t, s) = k(\boldsymbol{\tau}(t), \boldsymbol{\tau}(s))\, |d\boldsymbol{\tau}/ds|, \tag{6.7}$$

as well as the functions

$$\sigma(t) = \sigma(\boldsymbol{\tau}(t)) \quad \text{and} \quad f(t) = f(\boldsymbol{\tau}(t)).$$

Then techniques for solving

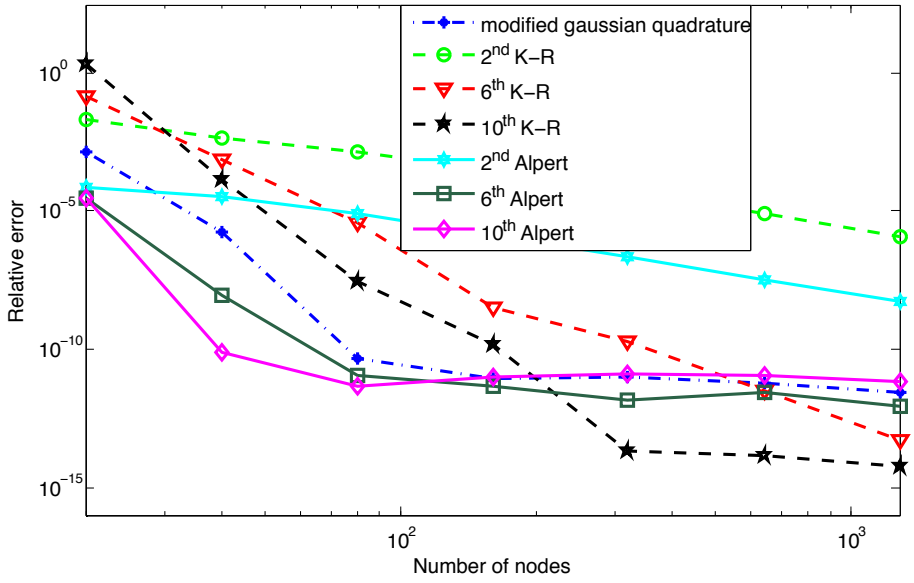
$$-\frac{1}{2}\sigma(t) + \int_0^T m(t, s)\, \sigma(s)\, ds = f(s), \qquad t \in [0, T], \tag{6.8}$$

will be equally applicable to (6.5).

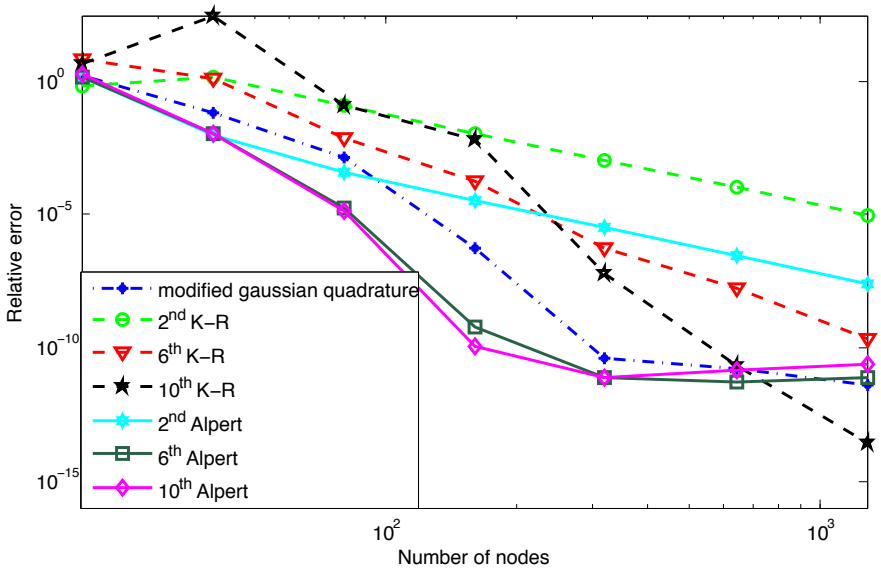
6.2.2. *Numerical results.* The geometries investigated are described in Figure 3. Exact solutions were generated by placing a few random point charges outside the domain. The solutions were evaluated at points defined on a circle encompassed by the boundary. The relative errors presented in Figures 4(a) - 4(b) were measured as in Section 6.1.

(a) The "circle" domain. (b) The "starfish" domain.

FIGURE 3. Contours used in numerical examples in Section 6.2 and 6.3. For the interior Laplace problem, the red dots inside represent the positions where the solutions were evaluated. Points outside the domain denote the point charges where the exact solutions were generated. For the exterior Helmholtz problem, the positions of the point charges and the positions where the exact solution was evaluated were reversed.



(a) Circular domain



(b) Starfish shaped domain

FIGURE 4. Error results for the interior Laplace problem (6.3) in Section 6.2 solved on domains (a) and (b) in Figure 3, respectively.

6.3. **Combined field discretization of the Helmholtz equation.** In this section, we solve exterior Dirichlet problem for the Helmholtz equation

$$-\Delta u - k^2 u = 0, \qquad \text{in } \ E = \Omega^c, \tag{6.9}$$

$$u = f, \qquad \text{on } \ \Gamma, \tag{6.10}$$

where $k > 0$ is the wavenumber. u satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - ik\,u\right) = 0, \tag{6.11}$$

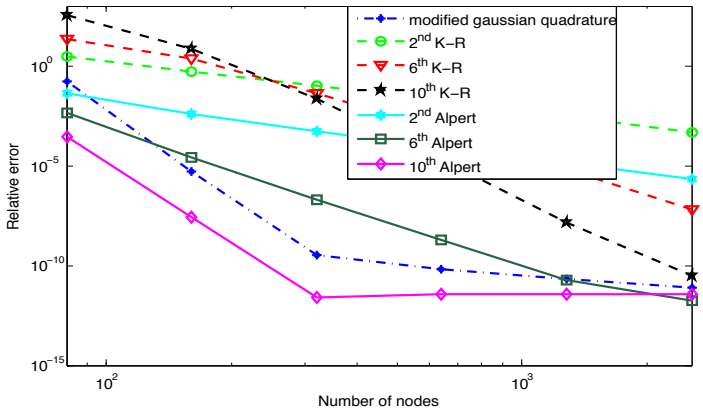
where $r = |\boldsymbol{x}|$ and the limit holds uniformly in all directions. The geometries are the same as in the previous example, except that in this case, we solved the potential in the domain exterior to Ω. A common solution to this is to represent the solution to (6.9) involving both the single and double layer acoustic potentials,

$$\begin{aligned} u(\boldsymbol{x}) &= \int_\Gamma k(\boldsymbol{x},\boldsymbol{x}')\,\sigma(\boldsymbol{x}')\,dl(\boldsymbol{x}') \\ &= \int_\Gamma \left(\frac{\partial \phi(\boldsymbol{x},\boldsymbol{x}')}{\partial \boldsymbol{n}(\boldsymbol{x}')} - ik\,\phi(\boldsymbol{x},\boldsymbol{x}')\right)\sigma(\boldsymbol{x}')\,dl(\boldsymbol{x}'), \qquad \boldsymbol{x}\in E, \end{aligned} \tag{6.12}$$

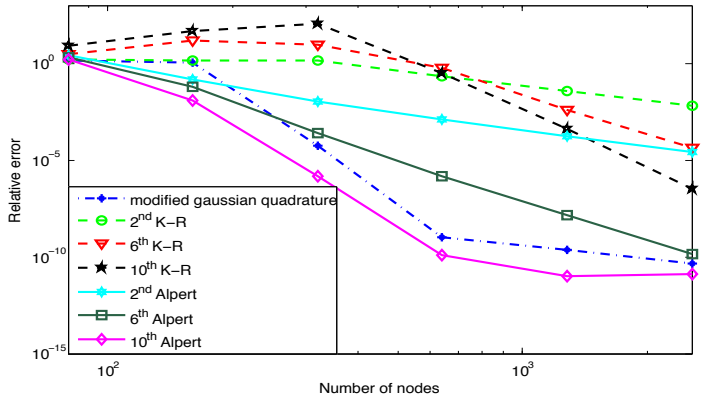
where $\phi(\boldsymbol{x},\boldsymbol{x}') = \frac{i}{4}H_0^{(1)}(k|\boldsymbol{x}-\boldsymbol{x}'|)$ and $H_0^{(1)}$ is the Hankel function of the first kind of order zero; $\boldsymbol{n}$ is the normal vector pointing outward to Γ. The motivation for the two-term representation (6.12) of the solution u is to obtain the unique solvability to problem (6.9). The corresponding boundary integral equation we need to solve is given by

$$\frac{1}{2}\sigma(\boldsymbol{x}) + \int_\Gamma k(\boldsymbol{x},\boldsymbol{x}')\sigma(\boldsymbol{x}')\,dl(\boldsymbol{x}') = f(\boldsymbol{x}), \qquad \boldsymbol{x}\in\Gamma, \tag{6.13}$$

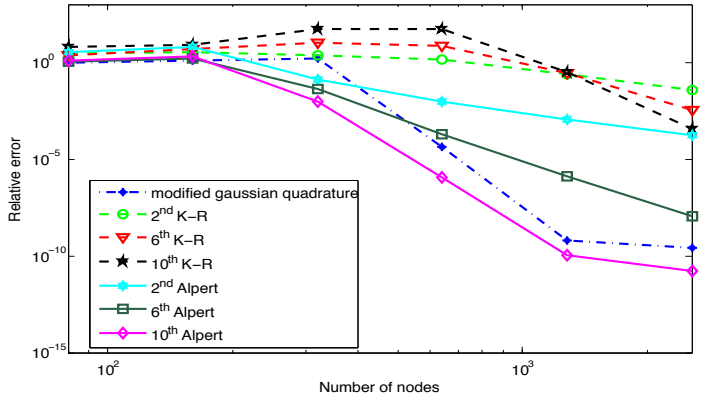
where $k(\boldsymbol{x},\boldsymbol{x}') = \frac{\partial\phi(\boldsymbol{x},\boldsymbol{x}')}{\partial\boldsymbol{n}(\boldsymbol{x}')} - ik\,\phi(\boldsymbol{x},\boldsymbol{x}')$. We have assessed the accuracy of each quadrature rule for two different domains, numbers of discretization nodes, and wavenumbers. Specifically, we varied wavenumbers such that there are 10, 25 and 50 wavelengths along the major axis in each domain. Relative errors of each quadrature are shown in Figure 5(a) - 6(c).



(a) 10 wavelengths in length (the major axis)

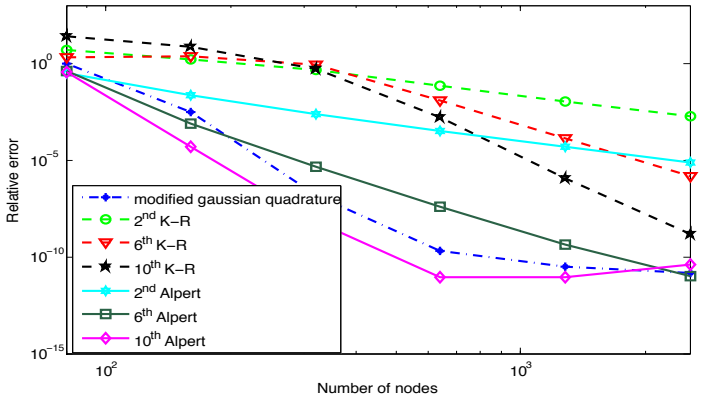


(b) 25 wavelengths in length (the major axis)

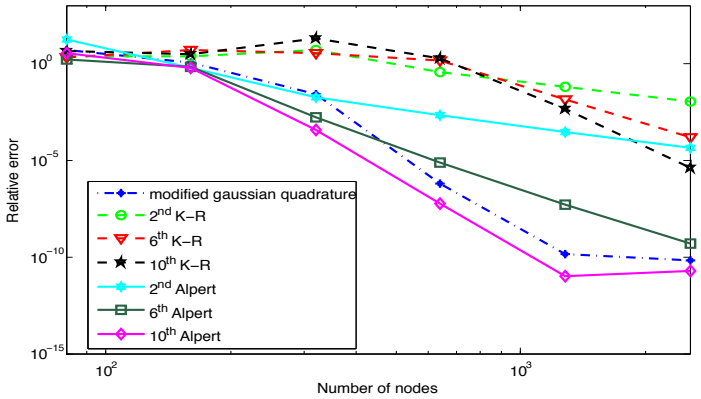


(c) 50 wavelengths in length (the major axis)

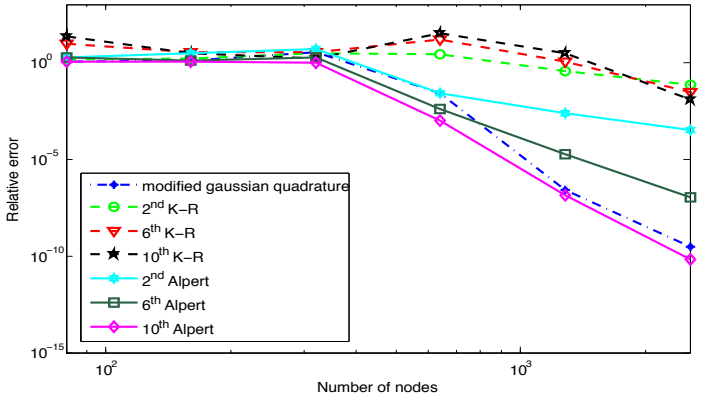
FIGURE 5. Error results for the exterior Helmholtz problem (6.9) in Section 6.3 solved on domain (a) in Figure 3.



(a) 10 wavelengths in length (the major axis)



(b) 25 wavelengths in length (the major axis)



(c) 50 wavelengths in length (the major axis)

FIGURE 6. Error results for the exterior Helmholtz problem (6.9) in Section 6.3 solved on domain (b) in Figure 3.

6.4. **The kernel associated with BIE on axisymmetric surfaces in $\mathbb{R}^3$.** In this section, we compare quadratures rules applied on kernels associated with BIEs on rotationally symmetric surfaces in $\mathbb{R}^3$. Specifically, we considered integral equations of the form

$$\sigma(\boldsymbol{x}) + \int_\Gamma k(\boldsymbol{x}, \boldsymbol{x}')\, \sigma(\boldsymbol{x}')\, dA(\boldsymbol{x}') = f(\boldsymbol{x}), \quad x \in \Gamma, \tag{6.14}$$

under the assumptions that Γ is a surface in $\mathbb{R}^3$ obtained by rotating a curve γ about an axis and the kernel function k is invariant under rotation about the symmetry axis. Figure 7 depicts domains used in numerical examples: the generating curves γ are shown in the left figures and the axisymmetric surfaces Γ are shown in the right ones. The BIE (6.14) on rotationally symmetric surfaces can via a Fourier transform be recast as a sequence of equations defined on the generating curve in cylindrical coordinates, i.e.

$$\sigma_n(r,z) + \sqrt{2\pi} \int_\Gamma k_n(r,z,r',z')\, \sigma_n(r',z')\, r'\, dl(r',z') = f_n(r,z), \quad (r,z) \in \gamma, \quad n \in \mathbb{Z}, \tag{6.15}$$

where σ_n, f_n, and k_n denote the Fourier coefficients of σ, f, and k, respectively. Details on how to truncate the Fourier series and construct the coefficient matrices for Laplace problem and Helmholtz problem can be found in [14]. In the following experiments, we consider BIE (6.14) was generated from interior Dirichlet Laplace problem, in which case $k(\boldsymbol{x}, \boldsymbol{x}')$ has expression

$$k(\boldsymbol{x}, \boldsymbol{x}') = \frac{\boldsymbol{n}(\boldsymbol{x}') \cdot (\boldsymbol{x} - \boldsymbol{x}')}{4\pi |\boldsymbol{x} - \boldsymbol{x}'|^3}. \tag{6.16}$$

As we recast the BIE defined on Γ to a sequence of equations defined on the generating curve γ, it is easy to see that the kernel function k_n has logarithmic singularities as $(r', z') \to (r, z)$ where we applied the corrected quadrature rules to maintain full accuracy. In this experiment, 101 Fourier modes were used and results are presented in Table 1 and 2.

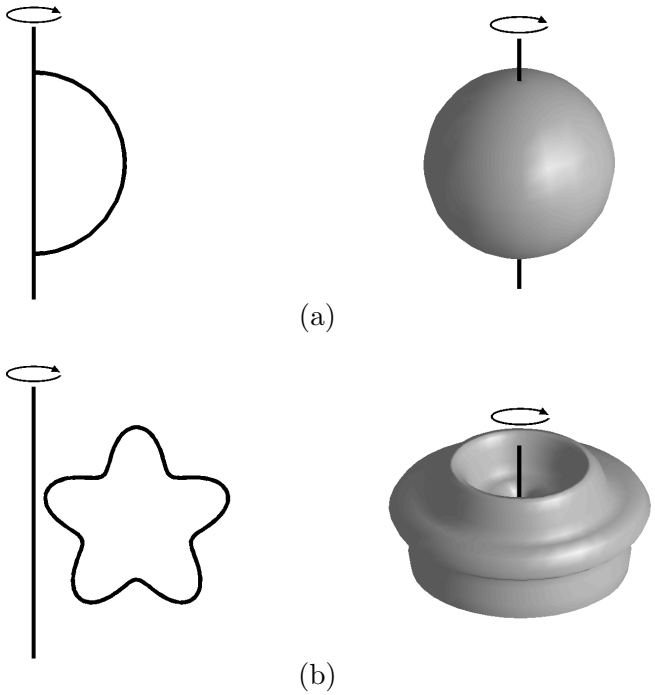


FIGURE 7. Domains used in numerical examples in Section 6.4. All items are rotated about the vertical axis. (a) A sphere. (b) A starfish torus.

TABLE 1. Error results for the interior Dirichlet Laplace problem in Section 6.4 solved on domain (a) in Figure 7 .

N	modified Gaussian	2^{nd}K-R	6^{th}K-R	10^{th}K-R	2^{nd}Alpert	6^{th}Alpert	10^{th}Alpert
20	3.169E-07	6.454E-03	1.089E-02	6.745E-02	1.375E-04	4.532E-04	2.084E-03
40	2.089E-11	2.965E-02	3.451E-03	8.121E-03	1.719E-05	8.425E-05	1.206E-04
80	2.737E-12	4.093E-03	1.470E-03	3.104E-05	1.592E-06	7.458E-06	1.574E-05
160	4.051E-13	5.640E-04	2.257E-04	3.429E-05	1.612E-07	1.131E-06	2.004E-06
320	3.287E-13	3.950E-05	5.136E-04	1.074E-05	1.774E-08	1.132E-07	2.331E-07

TABLE 2. Error results for the interior Dirichlet Laplace problem in Section 6.4 solved on domain (b) in Figure 7.

N	modified Gaussian	2^{nd}K-R	6^{th}K-R	10^{th}K-R	2^{nd}Alpert	6^{th}Alpert	10^{th}Alpert
20	3.672E-02	4.396E-02	1.651E-01	1.892E-01	5.990E-04	9.172E-04	2.148E-03
40	2.754E-04	2.139E-02	3.077E-02	1.758E-01	2.603E-04	1.891E-05	2.098E-04
80	2.174E-06	7.819E-03	3.306E-03	6.371E-05	4.387E-05	6.801E-08	4.928E-08
160	4.706E-10	1.552E-03	2.790E-06	4.390E-07	6.810E-06	1.456E-10	2.886E-11
320	1.120E-12	2.458E-04	2.195E-08	1.127E-10	1.016E-06	1.1885E-12	2.508E-12

7. Concluding remarks

In this section, we make some informal remarks on the benefits and drawbacks of the different quadrature rules. These are informed primarily by the numerical experiments in Section 6.

Kapur-Rokhlin quadrature: This rule is very efficient for solving BIEs defined on one-dimensional interval, as well as Laplace Dirichlet problem and Helmholtz problem with small wavenumbers. 14 digits of accuracy is obtained with 1 280 degrees of freedom when 10^{th}-order Kapur-Roklin rule applied on Laplace Dirichlet problem. However, the performance deteriorates as the wavenumber is increased. For instance, the 10^{th}-order Kapur-Rokhlin rule is completely inaccurate for Helmholtz problem with 50 wavelengths along the main axis of the domain, while modified Gaussian quadrature and 10^{th}-order Alpert quadrature are able to achieve 10 digits of accuracy with 2 560 discretization nodes. Another drawback is that it requires some quadrature nodes to lie outside the interval of integration, causing problems when integrating along open curves or on non-smooth domains. On the other hand, it is the simplest method to implement since no interpolation is needed.

Modified Gaussian quadrature: This rule is the most stable and adaptive method among the three in the sense of solving both low-frequency and high-frequency problems. In both low-frequency and high-frequency cases, 10 digits of accuracy has been obtained by placing at most 2 560 discretization nodes on the interval of integration. Further, there is no restriction on the interval of integration, saying that modified Gaussian quadrature is able to integrate singular functions over both closed and open contours very well. A drawback of this technique is that due to the crowding of Gaussian nodes near the ends of each panel, round-off errors make it difficult to attain accuracy close to the machine precision unless extended precision etc. in used in the generating the modified matrix elements. In our simplistic implementation, we never got more than ten or eleven correct digits no matter how many discretization points were used.

Alpert quadrature: This rule converges extremely fast to solve one-dimensional integral equation and BIEs associated with Laplace equation, as well as BIEs with low-frequency Helmholtz equation. In all the above cases, the 6^{th}-order Alpert quadrature converges faster than the modified Gaussian rule. However, for highly oscillatory functions, the 10^{th}-order Alpert quadrature has no distinguishable advantage over modified Gaussian quadrature. A main drawback of Alpert quadrature is, similar to Kapur-Rokhlin quadrature, it loses some accuracy when used for open contours.

For simplicity, we in this note limited our attention to the case of smooth contours, but both the Alpert and the modified Gaussian rule can with certain modifications be applied to contours with corners, see, e.g., [13, 5, 6, 4, 9].

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Appendix A. Tables of Modified Gaussian Quadrature Nodes and Weights

10 Point Gauss-Legendre Rule for integrals of the form $\int_{-1}^{1} f(x)\,dx$	
NODES	WEIGHTS
-9.739065285171716e-01	6.667134430868814e-02
-8.650633666889845e-01	1.494513491505806e-01
-6.794095682990244e-01	2.190863625159820e-01
-4.333953941292472e-01	2.692667193099963e-01
-1.488743389816312e-01	2.955242247147529e-01
1.488743389816312e-01	2.955242247147529e-01
4.333953941292472e-01	2.692667193099963e-01
6.794095682990244e-01	2.190863625159820e-01
8.650633666889845e-01	1.494513491505806e-01
9.739065285171716e-01	6.667134430868814e-02

20 point quadrature rule for integrals of the form $\int_{-1}^{1} f(x) + g(x)\log\lvert x_1 - x\rvert\,dx$, where x_1 is a Gauss-Legendre node	
NODES	WEIGHTS
-9.981629455677877e-01	4.550772157144354e-03
-9.915520723139890e-01	8.062764683328619e-03
-9.832812993252168e-01	7.845621096866406e-03
-9.767801773920733e-01	4.375212351185101e-03
-9.717169387169078e-01	1.021414662954223e-02
-9.510630103726074e-01	3.157199356768625e-02
-9.075765988474132e-01	5.592493151946541e-02
-8.382582352569804e-01	8.310260847601852e-02
-7.408522006801963e-01	1.118164522164500e-01
-6.147619568252419e-01	1.401105427713687e-01
-4.615244999958006e-01	1.657233639623953e-01
-2.849772954295424e-01	1.863566566231937e-01
-9.117593460489747e-02	1.999093145144455e-01
1.119089520342051e-01	2.046841584582030e-01
3.148842536644393e-01	1.995580161940930e-01
5.075733846631832e-01	1.841025430283230e-01
6.797470718157004e-01	1.586456191174843e-01
8.218833662202629e-01	1.242680229936124e-01
9.258924858821892e-01	8.273794370795576e-02
9.857595961761246e-01	3.643931593123844e-02

20 point quadrature rule for integrals of the form $\int_{-1}^{1} f(x) + g(x)\log\lvert x_2 - x\rvert\,dx$, where x_2 is a Gauss-Legendre node	
NODES	WEIGHTS
-9.954896691005256e-01	1.141744473788874e-02
-9.775532683688947e-01	2.368593568061651e-02
-9.500346715183706e-01	3.027205199814611e-02
-9.192373372373420e-01	3.021809354380292e-02
-8.916563772395616e-01	2.397183723558556e-02
-8.727728136507039e-01	1.253574079839078e-02
-8.607963163061316e-01	2.070840476545303e-02
-8.201318720954396e-01	6.080709508468810e-02
-7.394732321355052e-01	1.002402801599464e-01
-6.204853512352519e-01	1.371499151597280e-01
-4.667290485167077e-01	1.693838059093582e-01
-2.840823320902124e-01	1.945292086962893e-01
-8.079364608026202e-02	2.103223087093422e-01
1.328455136645940e-01	2.149900928447852e-01
3.451233500669768e-01	2.074984762344433e-01
5.437321547508867e-01	1.877085225595498e-01
7.167077216635750e-01	1.564543949958065e-01
8.534299232009863e-01	1.156104890379952e-01
9.458275339169444e-01	6.859369195724087e-02
9.912353127269481e-01	2.390220989094312e-02

20 point quadrature rule for integrals of the form $\int_{-1}^{1} f(x) + g(x)\log\lvert x_3 - x\rvert\,dx$, where x_3 is a Gauss-Legendre node	
NODES	WEIGHTS
-9.930122613589740e-01	1.779185041193254e-02
-9.643941806993207e-01	3.870503119897836e-02
-9.175869559770760e-01	5.371120494602663e-02
-8.596474181980754e-01	6.073467932536858e-02
-7.990442708271941e-01	5.901993373645797e-02
-7.443700671611690e-01	4.905519963921684e-02
-7.031684479828371e-01	3.249237036645046e-02
-6.811221147275545e-01	1.335394660596527e-02
-6.579449960254029e-01	4.151626407911676e-02
-5.949471688137100e-01	8.451456165895121e-02
-4.893032793226841e-01	1.262522607368499e-01
-3.441659232382107e-01	1.628408264966550e-01
-1.665388322404095e-01	1.907085686614375e-01
3.344207582228461e-02	2.071802230953481e-01
2.434356263087524e-01	2.105274833603497e-01
4.498696863725133e-01	2.000282912446872e-01
6.389777518528792e-01	1.760212445284564e-01
7.978632877793501e-01	1.399000904426490e-01
9.155180703268415e-01	9.402669072995991e-02
9.837258757826489e-01	4.161927873514264e-02

20 point quadrature rule for integrals of the form $\int_{-1}^{1} f(x) + g(x) \log \lvert x_4 - x \rvert \, dx$, where x_4 is a Gauss-Legendre node	
NODES	WEIGHTS
-9.903478871133073e-01	2.462513260640712e-02
-9.504025146897784e-01	5.449201732062665e-02
-8.834986023815121e-01	7.799498604905293e-02
-7.974523551287549e-01	9.241688894090601e-02
-7.022255002503461e-01	9.619882322938848e-02
-6.087194789244920e-01	8.902783806614303e-02
-5.275278952351541e-01	7.181973054766198e-02
-4.677586540799037e-01	4.663017060126023e-02
-4.360689210457623e-01	1.794303974050253e-02
-4.121945474875853e-01	4.061799823415495e-02
-3.494226766911471e-01	8.507517518447759e-02
-2.425993523586304e-01	1.277525783357134e-01
-9.646839923908594e-02	1.628510773009247e-01
7.921243716767302e-02	1.863323765408308e-01
2.715178194484646e-01	1.958227701927855e-01
4.658440358656903e-01	1.903138548150517e-01
6.472213975763533e-01	1.700731513381802e-01
8.015601619414859e-01	1.365784674773513e-01
9.168056007307982e-01	9.239595239693155e-02
9.839468743284722e-01	4.103797108164931e-02

20 point quadrature rule for integrals of the form $\int_{-1}^{1} f(x) + g(x) \log \lvert x_5 - x \rvert \, dx$, where x_5 is a Gauss-Legendre node	
NODES	WEIGHTS
-9.883561797860961e-01	2.974603958509255e-02
-9.398305159297058e-01	6.657945456889164e-02
-8.572399919019390e-01	9.731775484182564e-02
-7.482086250804679e-01	1.190433988432928e-01
-6.228514167093102e-01	1.297088242013777e-01
-4.928317114329241e-01	1.282900896966494e-01
-3.702771193724617e-01	1.148917968875341e-01
-2.666412108172461e-01	9.074932908233864e-02
-1.916083010783277e-01	5.818196361216740e-02
-1.521937160593461e-01	2.224697059733435e-02
-1.233125650067164e-01	4.788826761346366e-02
-5.257959675044444e-02	9.237500180593534e-02
5.877314311857769e-02	1.287410543031414e-01
2.012559739993003e-01	1.541960911507042e-01
3.627988191760868e-01	1.665885274544506e-01
5.297121321076323e-01	1.648585116745725e-01
6.878399330187783e-01	1.491408089644010e-01
8.237603202215137e-01	1.207592726093190e-01
9.259297297557394e-01	8.212177982524418e-02
9.856881498392895e-01	3.657506268226379e-02

20 point quadrature rule for integrals of the form $\int_{-1}^{1} f(x) + g(x) \log \lvert x_6 - x \rvert \, dx$, where x_6 is a Gauss-Legendre node	
NODES	WEIGHTS
-9.856881498392895e-01	3.657506268226379e-02
-9.259297297557394e-01	8.212177982524418e-02
-8.237603202215137e-01	1.207592726093190e-01
-6.878399330187783e-01	1.491408089644010e-01
-5.297121321076323e-01	1.648585116745725e-01
-3.627988191760868e-01	1.665885274544506e-01
-2.012559739993003e-01	1.541960911507042e-01
-5.877314311857769e-02	1.287410543031414e-01
5.257959675044444e-02	9.237500180593534e-02
1.233125650067164e-01	4.788826761346366e-02
1.521937160593461e-01	2.224697059733435e-02
1.916083010783277e-01	5.818196361216740e-02
2.666412108172461e-01	9.074932908233864e-02
3.702771193724617e-01	1.148917968875341e-01
4.928317114329241e-01	1.282900896966494e-01
6.228514167093102e-01	1.297088242013777e-01
7.482086250804679e-01	1.190433988432928e-01
8.572399919019390e-01	9.731775484182564e-02
9.398305159297058e-01	6.657945456889164e-02
9.883561797860961e-01	2.974603958509255e-02

20 point quadrature rule for integrals of the form $\int_{-1}^{1} f(x) + g(x) \log \lvert x_7 - x \rvert \, dx$, where x_7 is a Gauss-Legendre node	
NODES	WEIGHTS
-9.839468743284722e-01	4.103797108164931e-02
-9.168056007307982e-01	9.239595239693155e-02
-8.015601619414859e-01	1.365784674773513e-01
-6.472213975763533e-01	1.700731513381802e-01
-4.658440358656903e-01	1.903138548150517e-01
-2.715178194484646e-01	1.958227701927855e-01
-7.921243716767302e-02	1.863323765408308e-01
9.646839923908594e-02	1.628510773009247e-01
2.425993523586304e-01	1.277525783357134e-01
3.494226766911471e-01	8.507517518447759e-02
4.121945474875853e-01	4.061799823415495e-02
4.360689210457623e-01	1.794303974050253e-02
4.677586540799037e-01	4.663017060126023e-02
5.275278952351541e-01	7.181973054766198e-02
6.087194789244920e-01	8.902783806614303e-02
7.022255002503461e-01	9.619882322938848e-02
7.974523551287549e-01	9.241688894090601e-02
8.834986023815121e-01	7.799498604905293e-02
9.504025146897784e-01	5.449201732062665e-02
9.903478871133073e-01	2.462513260640712e-02

20 point quadrature rule for integrals of the form $\int_{-1}^{1} f(x) + g(x) \log \lvert x_8 - x \rvert \, dx$, where x_8 is a Gauss-Legendre node	
NODES	WEIGHTS
-9.837258757826489e-01	4.161927873514264e-02
-9.155180703268415e-01	9.402669072995991e-02
-7.978632877793501e-01	1.399000904426490e-01
-6.389777518528792e-01	1.760212445284564e-01
-4.498696863725133e-01	2.000282912446872e-01
-2.434356263087524e-01	2.105274833603497e-01
-3.344207582228461e-02	2.071802230953481e-01
1.665388322404095e-01	1.907085686614375e-01
3.441659232382107e-01	1.628408264966550e-01
4.893032793226841e-01	1.262522607368499e-01
5.949471688137100e-01	8.451456165895121e-02
6.579449960254029e-01	4.151626407911676e-02
6.811221147275545e-01	1.335394660596527e-02
7.031684479828371e-01	3.249237036645046e-02
7.443700671611690e-01	4.905519963921684e-02
7.990442708271941e-01	5.901993373645797e-02
8.596474181980754e-01	6.073467932536858e-02
9.175869559770760e-01	5.371120494602663e-02
9.643941806993207e-01	3.870503119897836e-02
9.930122613589740e-01	1.779185041193254e-02

20 point quadrature rule for integrals of the form $\int_{-1}^{1} f(x) + g(x) \log \lvert x_9 - x \rvert \, dx$, where x_9 is a Gauss-Legendre node	
NODES	WEIGHTS
-9.912353127269481e-01	2.390220989094312e-02
-9.458275339169444e-01	6.859369195724087e-02
-8.534299232009863e-01	1.156104890379952e-01
-7.167077216635750e-01	1.564543949958065e-01
-5.437321547508867e-01	1.877085225595498e-01
-3.451233500669768e-01	2.074984762344433e-01
-1.328455136645940e-01	2.149900928447852e-01
8.079364608026202e-02	2.103223087093422e-01
2.840823320902124e-01	1.945292086962893e-01
4.667290485167077e-01	1.693838059093582e-01
6.204853512352519e-01	1.371499151597280e-01
7.394732321355052e-01	1.002402801599464e-01
8.201318720954396e-01	6.080709508468810e-02
8.607963163061316e-01	2.070840476545303e-02
8.727728136507039e-01	1.253574079839078e-02
8.916563772395616e-01	2.397183723558556e-02
9.192373372373420e-01	3.021809354380292e-02
9.500346715183706e-01	3.027205199814611e-02
9.775532683688947e-01	2.368593568061651e-02
9.954896691005256e-01	1.141744473788874e-02

20 point quadrature rule for integrals of the form $\int_{-1}^{1} f(x) + g(x) \log \lvert x_{10} - x \rvert \, dx$, where x_{10} is a Gauss-Legendre node	
NODES	WEIGHTS
-9.857595961761246e-01	3.643931593123844e-02
-9.258924858821892e-01	8.273794370795576e-02
-8.218833662202629e-01	1.242680229936124e-01
-6.797470718157004e-01	1.586456191174843e-01
-5.075733846631832e-01	1.841025430283230e-01
-3.148842536644393e-01	1.995580161940930e-01
-1.119089520342051e-01	2.046841584582030e-01
9.117593460489747e-02	1.999093145144455e-01
2.849772954295424e-01	1.863566566231937e-01
4.615244999958006e-01	1.657233639623953e-01
6.147619568252419e-01	1.401105427713687e-01
7.408522006801963e-01	1.118164522164500e-01
8.382582352569804e-01	8.310260847601852e-02
9.075765988474132e-01	5.592493151946541e-02
9.510630103726074e-01	3.157199356768625e-02
9.717169387169078e-01	1.021414662954223e-02
9.767801773920733e-01	4.375212351185101e-03
9.832812993252168e-01	7.845621096866406e-03
9.915520723139890e-01	8.062764683328619e-03
9.981629455677877e-01	4.550772157144354e-03

24 point quadrature rule for integrals of the form $\int_{0}^{1} f(x) + g(x) \log(x + \bar{x}) dx$, where $\bar{x} \geq 10^{-1}$	
NODES	WEIGHTS
3.916216329415252e-02	4.880755296918116e-02
8.135233983530081e-02	3.196002785163611e-02
1.123448211344994e-01	3.883416642507362e-02
1.595931983965030e-01	5.148898992140820e-02
2.085759027831349e-01	4.219328148763533e-02
2.426241962027560e-01	3.420686213633789e-02
2.886190312538522e-01	5.512488680719239e-02
3.469021762354675e-01	6.007112809843418e-02
4.072910101569611e-01	6.022350479415180e-02
4.664019722595442e-01	5.735022004401478e-02
5.182120817844112e-01	4.167923417118068e-02
5.501308436771654e-01	3.346089628879600e-02
5.970302980854608e-01	5.574716218423796e-02
6.548457960388209e-01	5.847838243344473e-02
7.119542126106005e-01	5.464156990092474e-02
7.607920420946340e-01	4.092186343704961e-02
7.953017051155684e-01	3.283728166050225e-02
8.303900341517088e-01	3.438233273473095e-02
8.612724919009394e-01	3.022585192226418e-02
8.954049128027080e-01	3.700769701277380e-02
9.315909369155358e-01	3.410213679365162e-02
9.621742249068356e-01	2.665791885274193e-02
9.843663446380599e-01	1.754420526360429e-02
9.970087425823398e-01	7.662283104388867e-03

24 point quadrature rule for integrals of the form $\int_0^1 f(x) + g(x)\log(x+\bar{x})dx$, where $10^{-2} \le \bar{x} \le 10^{-1}$	
NODES	WEIGHTS
1.940564616937581e-02	2.514022176052795e-02
4.545433992382339e-02	2.703526530535647e-02
7.378866604396420e-02	2.980872487617485e-02
1.054147718077606e-01	3.360626237885489e-02
1.412997888401000e-01	3.829678083416609e-02
1.822325567811081e-01	4.365651045780837e-02
2.287282121202408e-01	4.935846322319046e-02
2.809170925514041e-01	5.495967924055210e-02
3.384320962237970e-01	5.991162198705084e-02
4.003108031244078e-01	6.356960862248889e-02
4.648605571606025e-01	6.506868552467118e-02
5.290714994276687e-01	6.219588235225894e-02
5.829663557386375e-01	3.889986041695310e-02
6.128301889979477e-01	3.573431931940621e-02
6.606072156240962e-01	5.296315368353523e-02
7.139495966128518e-01	5.369033999927759e-02
7.677830914961244e-01	5.340793573367282e-02
8.187382423336450e-01	4.704756013998560e-02
8.587068551739496e-01	3.276576301747068e-02
8.906873285570645e-01	3.449175311880027e-02
9.267772492129903e-01	3.560168848238671e-02
9.592137652582382e-01	2.857367151127661e-02
9.830962712794008e-01	1.894042942442201e-02
9.967621546194148e-01	8.291994770212826e-03

24 point quadrature rule for integrals of the form $\int_0^1 f(x) + g(x)\log(x+\bar{x})dx$, where $10^{-3} \le \bar{x} \le 10^{-2}$	
NODES	WEIGHTS
7.571097817272427e-03	9.878088201321919e-03
1.800655325976786e-02	1.109316819462674e-02
3.003901004577040e-02	1.313311581321880e-02
4.462882147989575e-02	1.624262442061470e-02
6.295732618092606e-02	2.065168462990214e-02
8.644035241970913e-02	2.657795406825320e-02
1.166164809306920e-01	3.399052299072427e-02
1.546690628394902e-01	4.208214612865170e-02
1.999554346680615e-01	4.732516974042797e-02
2.434683359132119e-01	3.618419415803922e-02
2.800846274146029e-01	4.547346840583578e-02
3.368595257878888e-01	6.463153575242817e-02
4.044418359833648e-01	6.859104457897808e-02
4.685002493634456e-01	5.589917935916451e-02
5.185062817085154e-01	5.199232318335285e-02
5.811314144990846e-01	7.089840644422261e-02
6.545700991450585e-01	7.427400331494240e-02
7.276588861478224e-01	7.125308736931726e-02
7.960626077582168e-01	6.513697474660338e-02
8.572037183403355e-01	5.682298546820264e-02
9.091330485015775e-01	4.678000924507099e-02
9.503131649503738e-01	3.538488886617123e-02
9.795718963793163e-01	2.299723483013955e-02
9.961006479199827e-01	9.993597414733579e-03

24 point quadrature rule for integrals of the form $\int_0^1 f(x) + g(x)\log(x+\bar{x})dx$, where $10^{-4} \le \bar{x} \le 10^{-3}$	
NODES	WEIGHTS
2.625961371586153e-03	3.441901737135120e-03
6.309383772392260e-03	3.978799794732070e-03
1.073246133489697e-02	4.958449505644980e-03
1.645170499644402e-02	6.620822501994994e-03
2.433800511777796e-02	9.385496468197222e-03
3.582530925992294e-02	1.396512052439178e-02
5.315827372101662e-02	2.119383832447796e-02
7.917327903614484e-02	3.124989308824302e-02
1.162053707416708e-01	4.291481168916344e-02
1.648139164451449e-01	5.400832278279924e-02
2.231934088488800e-01	6.197424674301215e-02
2.864519293820641e-01	6.297221626131570e-02
3.466729491189400e-01	5.794981636764223e-02
4.076175535528108e-01	6.650501614478806e-02
4.800964107543535e-01	7.716379373230733e-02
5.594105009204460e-01	8.047814122759604e-02
6.395390292352857e-01	7.917822434973971e-02
7.167410782176877e-01	7.477646096014055e-02
7.882807127957939e-01	6.793424765652059e-02
8.519356675821297e-01	5.906852968947303e-02
9.058606177202579e-01	4.853108558910315e-02
9.485539755760567e-01	3.666228059710319e-02
9.788566874094059e-01	2.380850649522536e-02
9.959649506960162e-01	1.034186239262945e-02

24 point quadrature rule for integrals of the form $\int_0^1 f(x) + g(x)\log(x+\bar{x})dx$, where $10^{-5} \le \bar{x} \le 10^{-4}$	
NODES	WEIGHTS
7.759451679242260e-04	1.049591733965263e-03
1.952854410117286e-03	1.314968855711329e-03
3.429053832116395e-03	1.651475072547296e-03
5.301128540262913e-03	2.135645684467029e-03
7.878118775220067e-03	3.165043382856636e-03
1.205537050949829e-02	5.479528688655274e-03
1.965871512055557e-02	1.028817002915096e-02
3.403328641997047e-02	1.923291785614007e-02
5.947430305925957e-02	3.212643438782854e-02
9.873500543531440e-02	4.638626850049229e-02
1.518862681939413e-01	5.960676923068444e-02
2.171724325134259e-01	7.052360405410943e-02
2.919941878735093e-01	7.863451090237836e-02
3.734637353255530e-01	8.381771698595157e-02
4.586710018443288e-01	8.612755554083525e-02
5.448057416999684e-01	8.569938467103264e-02
6.292158981939618e-01	8.271051499695768e-02
7.094415843889587e-01	7.736692567834522e-02
7.832417328632321e-01	6.990012937760461e-02
8.486194141302759e-01	6.056687669667680e-02
9.038469149367938e-01	4.964868706783169e-02
9.474898150194623e-01	3.745026957972177e-02
9.784290662963747e-01	2.429741981889855e-02
9.958843370550371e-01	1.054906616108520e-02

24 point quadrature rule for integrals of the form $\int_0^1 f(x)+g(x)\log(x+\bar{x})dx$, where $10^{-6}\le\bar{x}\le 10^{-5}$	
NODES	WEIGHTS
3.126377187332637e-04	4.136479682893960e-04
7.671264269072188e-04	5.068714387414649e-04
1.359575160544077e-03	7.008932527842778e-04
2.238313285727558e-03	1.110264922990352e-03
3.770276623583326e-03	2.120108385941761e-03
7.146583956092048e-03	5.249076343206215e-03
1.635515250548719e-02	1.450809938905405e-02
3.828062855101241e-02	2.987724029376343e-02
7.628984500206759e-02	4.593298717863718e-02
1.294255336121595e-01	5.987634475538021e-02
1.949876755761554e-01	7.065953519392547e-02
2.693852297828856e-01	7.729918562776261e-02
3.469762441631538e-01	7.556635340171830e-02
4.122748928895491e-01	5.234123638339037e-02
4.662499202239145e-01	6.532130125393047e-02
5.421402737123784e-01	8.188272080198840e-02
6.248832413655412e-01	8.237354882288161e-02
7.053258496784840e-01	7.795795664563893e-02
7.798841313231049e-01	7.076514272025076e-02
8.461534275163378e-01	6.145788741452406e-02
9.022312524979976e-01	5.044339641339403e-02
9.465899812310277e-01	3.807817118430632e-02
9.780549563823810e-01	2.471549011101626e-02
9.958125149101927e-01	1.073289672726758e-02

24 point quadrature rule for integrals of the form $\int_0^1 f(x)+g(x)\log(x+\bar{x})dx$, where $10^{-7}\le\bar{x}\le 10^{-6}$	
NODES	WEIGHTS
1.019234906342863e-04	1.349775051746596e-04
2.506087227631447e-04	1.663411550150506e-04
4.461429005344285e-04	2.328782111562424e-04
7.422845421202523e-04	3.804721779784063e-04
1.289196091156456e-03	7.930350452911450e-04
2.739287668024851e-03	2.600694722423854e-03
9.075168969969708e-03	1.212249113599252e-02
2.968005234555358e-02	2.946708975720586e-02
6.781742979962609e-02	4.647771960691390e-02
1.217792474402805e-01	6.095376889009233e-02
1.886625378438471e-01	7.224844725827559e-02
2.650602155844836e-01	7.986429603884565e-02
3.465113608339080e-01	8.143206462900546e-02
4.178374197420536e-01	5.040529357007135e-02
4.597624982511183e-01	5.592137651001418e-02
5.348065111487157e-01	8.398073572656715e-02
6.194640153146728e-01	8.402586870225486e-02
7.013481004172354e-01	7.922223490159952e-02
7.770386175609082e-01	7.177919251691964e-02
8.442211768916794e-01	6.227551999401272e-02
9.010272836291835e-01	5.108407212719758e-02
9.459409782755001e-01	3.854783279333592e-02
9.777905486554876e-01	2.501496650831813e-02
9.957622871041650e-01	1.086176801402067e-02

24 point quadrature rule for integrals of the form $\int_0^1 f(x)+g(x)\log(x+\bar{x})dx$, where $10^{-8}\le\bar{x}\le 10^{-7}$	
NODES	WEIGHTS
3.421721832247593e-05	4.559730842497453e-05
8.533906255442380e-05	5.840391255974745e-05
1.563524616155011e-04	8.761580900682040e-05
2.746612401575526e-04	1.617264666294872e-04
5.408643931265062e-04	4.433543035169213e-04
1.782382096488333e-03	3.116175111368442e-03
1.101243912052365e-02	1.655494413772595e-02
3.553172024884285e-02	3.242539256461602e-02
7.554170435463801e-02	4.734426463929677e-02
1.295711894941649e-01	6.032614603579952e-02
1.953213037793089e-01	7.069975187373848e-02
2.699680545714222e-01	7.806973621204365e-02
3.503697281371090e-01	8.216350598137868e-02
4.330838596494367e-01	8.261286657092808e-02
5.141801680435878e-01	7.883476216668445e-02
5.895097016206093e-01	7.157205125318401e-02
6.582708672338614e-01	6.703064468754417e-02
7.252543617887320e-01	6.706137273719630e-02
7.914154485613720e-01	6.449984116349734e-02
8.528383935857844e-01	5.775434959088197e-02
9.059696536862878e-01	4.812600239023880e-02
9.484664124578303e-01	3.661415869304224e-02
9.787863313133854e-01	2.386304203446463e-02
9.959482975155097e-01	1.038268695581411e-02

24 point quadrature rule for integrals of the form $\int_0^1 f(x)+g(x)\log(x+\bar{x})dx$, where $10^{-9}\le\bar{x}\le 10^{-8}$	
NODES	WEIGHTS
6.538987938840374e-06	1.500332421093607e-05
2.613485075847413e-05	2.367234654253158e-05
5.664183720634991e-05	4.007286246706405e-05
1.179374114362569e-04	9.497743501485505e-05
3.299119431334128e-04	4.619067037944727e-04
3.626828607577001e-03	9.985382463808036e-03
2.265102906572155e-02	2.805741744607257e-02
5.896796231680340e-02	4.404106103008398e-02
1.092496277855923e-01	5.548413172821072e-02
1.666701689499393e-01	5.693235996372726e-02
2.196889385898800e-01	5.087307376046002e-02
2.770352260035617e-01	6.593729718379782e-02
3.483163928268329e-01	7.335680008972614e-02
4.153287664837260e-01	5.675029500743735e-02
4.695624219668608e-01	6.117926027541254e-02
5.421129318998841e-01	8.004805067067550e-02
6.238832212055707e-01	8.196991767042605e-02
7.041842972237081e-01	7.800219127200407e-02
7.788817007552110e-01	7.097175077519494e-02
8.453877637047045e-01	6.171193295041172e-02
9.017178251963006e-01	5.068671319716005e-02
9.462999385952402e-01	3.827738423897266e-02
9.779333485180249e-01	2.485063762733620e-02
9.957890687155009e-01	1.079284973329516e-02

24 point quadrature rule for integrals of the form $\int_0^1 f(x) + g(x)\log(x+\bar{x})dx$, where $10^{-10} \leq \bar{x} \leq 10^{-9}$	
NODES	WEIGHTS
6.725520559705825e-06	8.128391913974039e-05
6.986424152770461e-06	-7.773900735768282e-05
1.217363416714366e-05	1.287386499666193e-05
2.677746219601529e-05	1.895577251914526e-05
5.597036348896741e-05	4.732580352158076e-05
2.729343280943077e-04	9.857909615386162e-04
9.445526806263141e-03	1.756872897270054e-02
3.556725025161542e-02	3.439422017906772e-02
7.765556668177810e-02	4.944188361792970e-02
1.336848150648662e-01	6.219733934997792e-02
2.011576917683550e-01	7.228007436918939e-02
2.772736854314979e-01	7.944986391225688e-02
3.590124362607926e-01	8.347646288178011e-02
4.430074035214462e-01	8.380433020121207e-02
5.247388219574510e-01	7.832768209682506e-02
5.961053238782420e-01	6.300796225242940e-02
6.547331131213409e-01	5.923406014585053e-02
7.192258519628951e-01	6.834293563803810e-02
7.874251789073102e-01	6.660337204499726e-02
8.505852012775045e-01	5.911988751082552e-02
9.047824617894323e-01	4.893575310568894e-02
9.479045131744448e-01	3.708256438629509e-02
9.785770588866582e-01	2.411463784693618e-02
9.959104692340199e-01	1.048087156697020e-02

24 point quadrature rule for integrals of the form $\int_0^1 f(x) + g(x)\log(x+\bar{x})dx$, where $10^{-11} \leq \bar{x} \leq 10^{-10}$	
NODES	WEIGHTS
2.828736694877886e-08	1.665602686704325e-05
2.302233157554212e-06	2.577419924039251e-06
5.853587143444178e-06	4.957941112780975e-06
1.451588770083244e-05	1.537074702915107e-05
9.711965099273031e-05	4.640075239797995e-04
9.004761967373848e-03	1.705687938176189e-02
3.442077924035546e-02	3.349724914160473e-02
7.543926781582543e-02	4.820210872119093e-02
1.300373356318913e-01	6.054547286337976e-02
1.955182772803384e-01	6.984354388121057e-02
2.683608546664295e-01	7.498721497014774e-02
3.430029178740901e-01	7.240620145057083e-02
4.085056107803621e-01	5.774925310174693e-02
4.660198270439085e-01	6.238505554837956e-02
5.336124745634699e-01	6.940394677081842e-02
5.985245800106473e-01	5.910843483407385e-02
6.564089719608276e-01	6.059752321454190e-02
7.216666024232565e-01	6.823362237770209e-02
7.893712241343741e-01	6.593839664071163e-02
8.518883782001418e-01	5.853014420243146e-02
9.055688088881344e-01	4.849217100974983e-02
9.483163097840529e-01	3.677417821170115e-02
9.787413692715607e-01	2.392585642844202e-02
9.959413203611228e-01	1.040149939671874e-02

24 point quadrature rule for integrals of the form $\int_0^1 f(x) + g(x)\log(x+\bar{x})dx$, where $10^{-12} \leq \bar{x} \leq 10^{-11}$	
NODES	WEIGHTS
6.147063879573664e-07	8.763741095000331e-07
2.102921984985835e-06	1.784696796288373e-05
2.188366117432289e-06	-1.795398395983826e-05
3.482602942694880e-06	5.117514567175025e-06
2.768001888608636e-05	1.698863549284390e-04
8.942779215792784e-03	1.701975216672032e-02
3.432218364237253e-02	3.346025972593909e-02
7.530931328026620e-02	4.817949622196712e-02
1.298983048592572e-01	6.055152664710045e-02
1.954020797117703e-01	6.988313730886592e-02
2.682970870436427e-01	7.504602275463067e-02
3.429540704041702e-01	7.230942674874111e-02
4.080399755202422e-01	5.705952259766429e-02
4.652562798154792e-01	6.265021180818162e-02
5.333220999210325e-01	6.993669694523695e-02
5.986982369433125e-01	5.937130986945129e-02
6.564773600603511e-01	6.026572020863567e-02
7.215159032030418e-01	6.815292696374753e-02
7.892098210760941e-01	6.596804590657802e-02
8.517672777806986e-01	5.857483758149194e-02
9.054906995605498e-01	4.853209199396977e-02
9.482736017320823e-01	3.680469214176019e-02
9.787238593479314e-01	2.394561701705853e-02
9.959379852805677e-01	1.041005152890511e-02

24 point quadrature rule for integrals of the form $\int_0^1 f(x) + g(x)\log(x+\bar{x})dx$, where $10^{-13} \leq \bar{x} \leq 10^{-12}$	
NODES	WEIGHTS
4.523740015216508e-08	4.418138082366788e-07
4.281855233588279e-07	4.389108058643120e-07
1.036900153156159e-06	9.539585150737866e-07
7.825849325746907e-06	5.823980947200484e-05
8.617419723953112e-03	1.634464263521301e-02
3.268881163637599e-02	3.129682188728318e-02
6.988441391437043e-02	4.212468617589480e-02
1.142202307676442e-01	4.505120897719191e-02
1.596471081833281e-01	4.769069780026684e-02
2.135336418959620e-01	6.038503382768951e-02
2.781100275296151e-01	6.695343672694180e-02
3.433392803364457e-01	6.163298712826237e-02
4.019960595528027e-01	5.877742624357513e-02
4.656415679416787e-01	6.800053637773440e-02
5.334880548894250e-01	6.516918103589647e-02
5.943298528903542e-01	5.853785375926075e-02
6.562968737815924e-01	6.639396325654251e-02
7.250343344601498e-01	6.948738324081696e-02
7.928820737781136e-01	6.538801703374268e-02
8.546103048745466e-01	5.761503751629250e-02
9.073762310762705e-01	4.761344859555310e-02
9.493253659835347e-01	3.607033097268266e-02
9.791606801267259e-01	2.345690720840071e-02
9.960217573957566e-01	1.019557402722854e-02

24 point quadrature rule for integrals of the form $\int_0^1 f(x) + g(x)\log(x+\bar{x})dx$, where $10^{-14} \leq \bar{x} \leq 10^{-13}$	
NODES	WEIGHTS
6.025980282801020e-08	9.079353616441234e-07
6.411245262925473e-08	-8.390389042773805e-07
1.862815529429129e-07	2.782460677485016e-07
2.029190208906422e-06	1.821115881362725e-05
8.902881307076499e-03	1.695809650660321e-02
3.420089035164912e-02	3.336370146025145e-02
7.508687525931594e-02	4.807898681796971e-02
1.295858123029775e-01	6.047672723211479e-02
1.950409815188335e-01	6.986774906175534e-02
2.679751967812604e-01	7.515608233194288e-02
3.428525062164689e-01	7.264249904037610e-02
4.080941369413548e-01	5.672507168477261e-02
4.646644511900009e-01	6.220316364524964e-02
5.328071517215501e-01	7.032362652293805e-02
5.978508749698001e-01	5.742730804758014e-02
6.521214523350964e-01	5.644075454541152e-02
7.134921670665336e-01	6.318643666150391e-02
7.679317896479284e-01	3.945995610428228e-02
8.029718487208403e-01	4.324200884758527e-02
8.551101435866935e-01	5.478223695609097e-02
9.067319102017767e-01	4.740856250832772e-02
9.487765213293372e-01	3.633314063504751e-02
9.788979796532736e-01	2.372788917088821e-02
9.959684838634199e-01	1.033036588606145e-02

Appendix B. Tables of Alpert Quadrature Nodes and Weights

2^{nd}-order Alpert Quadrature Rule for integrals of the form $\int_0^1 f(x) + g(x)\log(x)\,dx$, with $a = 1$	
NODES	WEIGHTS
1.591549430918953e-01	5.000000000000000e-01

6^{th}-order Alpert Quadrature Rule for integrals of the form $\int_0^1 f(x) + g(x)\log(x)\,dx$, with $a = 3$	
NODES	WEIGHTS
4.004884194926570e-03	1.671879691147102e-02
7.745655373336686e-02	1.636958371447360e-01
3.972849993523248e-01	4.981856569770637e-01
1.075673352915104e+00	8.372266245578912e+00
2.003796927111872e+00	9.841730844088381e+00

10^{th}-order Alpert Quadrature Rule for integrals of the form $\int_0^1 f(x) + g(x)\log(x)\,dx$, with $a = 6$	
NODES	WEIGHTS
1.175089381227308e-03	4.560746882084207e-03
1.877034129831289e-02	3.810606322384757e-02
9.686468391426860e-02	1.293864997289512e-01
3.004818668002884e-01	2.884360381408835e-01
6.901331557173356e-01	4.958111914344961e-01
1.293695738083659e+00	7.077154600594529e-01
2.090187729798780e+00	8.741924365285083e-01
3.016719313149212e+00	9.661361986515218e-01
4.001369747872486e+00	9.957887866078700e-01
5.000025661793423e+00	9.998665787423845e-01