- This exam is worth 100 points and has 6 problems. Do all problems.
- Show all work. Simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOU HAVE SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE!** (Failure to do so may result in a penalty or a score of $0/100$ on this exam.)
- You are taking this exam in a proctored and honor code enforced environment. Calculators, cell phones, or other electronic devices or the internet are not permitted during the exam. One 8.5"$\times$11" crib sheet with writing on one side is allowed.

---

0. At the top of the page containing your solution to problem 1, write the following statement: "I will abide by the CU Boulder Honor Code on this exam" and sign your name to it.

1. (10 pts) Write the word TRUE or FALSE as appropriate. *No work need be shown.* No partial credit given. Do not use the abbreviation "T" or "F", completely write out either "TRUE" or "FALSE" depending on your selection. (If any of your answers are hard to locate or difficult to read, points will be deducted.)

   (a) If A and B are invertible $n \times n$ matrices, then their sum $C = A + B$ must be invertible as well.

   (b) Let $A \in \mathbb{M}_{44}$. Then $|3A| = 3^4|A|$.

   (c) A set of vectors is linearly dependent if the zero vector belongs to their span.

   (d) If matrix A has all zeros on its main diagonal, then it is singular.

   (e) The vectors $\mathbf{v}_1 = \left[\begin{smallmatrix}2\\4\end{smallmatrix}\right]$, $\mathbf{v}_2 = \left[\begin{smallmatrix}-1\\-5\end{smallmatrix}\right]$, $\mathbf{v}_3 = \left[\begin{smallmatrix}-3\\-6\end{smallmatrix}\right]$ span $\mathbb{R}^2$.

   Solution:

   (a) FALSE Invertibility is **not** preserved under addition. As a counterexample, let $A = I_n$ and $B = -I_n$. Both are invertible, since $\det[A] = 1 \neq 0$ and $\det[B] = (-1)^n \neq 0$. However, their sum

   $$C = A + B = I_n + (-I_n) = \mathbf{0}_n$$

   is the zero matrix with $\det[C] = 0$ and is therefore a singular matrix, i.e. matrix $C = A + B$ is a non-invertible matrix.

   (b) TRUE Scaling a matrix multiplies *every* one of its four rows by 3, and each row scaling pulls out one factor of 3 from the determinant. Hence $|3A| = 3^4|A| = 81|A|$. In general $|cA| = c^n|A|$ for $A \in \mathbb{M}_{nn}$.

   (c) FALSE Note $\mathbf{0} \in \text{span}\{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\}$ since $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] = 0 \cdot \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + 0 \cdot \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ but clearly $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ are not linearly dependent.

   (d) FALSE Consider the counterexample $A = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$.

   (e) TRUE Putting the vectors in a matrix and row reducing we have $\left[\begin{smallmatrix}2&-1&-3\\4&-5&-6\end{smallmatrix}\right] \rightarrow \left[\begin{smallmatrix}2&-1&-3\\0&-3&0\end{smallmatrix}\right]$. The reduced matrix has two pivots so the vectors span $\mathbb{R}^2$.

2. (15 pts)

   Given $A = \begin{bmatrix}1 & 1 & -2\\ 3 & -1 & 0\\ 2 & 0 & 1\end{bmatrix}$, $B = \begin{bmatrix}2 & 0 & 0\\ -1 & 0 & 1\\ 1 & 1 & 2\end{bmatrix}$, and $\vec{c} = \begin{bmatrix}1\\2\\2\end{bmatrix}$, compute the following expressions or solve for $\vec{x}$.

   (a) AB

   (b) $(BA)^T(B^{-1})^T$

   (c) $(A + B)^{-1}\vec{x} = \vec{c}$

   Solution:

   (a) $AB = \begin{bmatrix}-1 & -2 & -3\\ 7 & 0 & -1\\ 5 & 1 & 2\end{bmatrix}$.

Copyright APPM Do not post

(b) $(BA)^T(B^{-1})^T = A^TB^T(B^T)^{-1} = A^T = \begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

(c) $(A+B)^{-1}\vec{x} = \vec{c} \implies \vec{x} = (A+B)\vec{c} = \begin{bmatrix} 3 & 1 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 11 \end{bmatrix}$.

3. (21 pts) The following problems are **not** related. *Do all problems.*

   (a) (6 pts) Use *row operations* to find the **reduced row echelon form** of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 6 \\ 5 & 8 & 12 \\ 0 & 8 & 14 \end{bmatrix}$. Show all steps.

   (b) (8 pts) Suppose A is a nonzero 3×4 matrix and $\mathbf{b}$ satisfies $A\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 2\mathbf{b}$. If $U = \begin{bmatrix} 1 & 1 & -3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is the *row echelon form* of A, find the general solution of $A\mathbf{x} = \mathbf{b}$. Justify your answer.

   (c) (7 pts) For what values of k does the augmented system $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ -1 & -2 & 2 & 1 & 3 \\ -1 & -2 & 7 & 6 & 2k \end{array}\right]$ have (i) no solution? (ii) infinitely many solutions? (iii) a unique solution? Explain.

Solution:

(a)(pts) Using elementary row operations, we have:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 6 \\ 5 & 8 & 12 \\ 0 & 8 & 14 \end{bmatrix} \xrightarrow[2.\ R_3^* = -5R_1 + R_3]{1.\ R_2^* = -2R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 6 \\ 0 & 8 & 12 \\ 0 & 8 & 14 \end{bmatrix} \xrightarrow[4.\ R_4^* = -2R_2 + R_4]{3.\ R_3^* = -2R_2 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{5.\ R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{6.\ R_2^* = -3R_3 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(b)(pts) By the Nonhomogeneous Principle, the general solution has the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ where $\mathbf{x}_p$ satisfies $A\mathbf{x}_p = \mathbf{b}$ and $\mathbf{x}_h$ is the general solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Particular solution We are given $A \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 2\mathbf{b}$, thus, by the linearity of the operator $L[\mathbf{x}] = A\mathbf{x}$, we have $A \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \mathbf{b}$ therefore a candidate for the particular solution is the vector $\mathbf{x}_p = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$.

Homogeneous solution Note that, using elementary row operations on the homogeneous system $[A|\mathbf{0}]$, we can conclude that

$$[A|\mathbf{0}] \ \rightarrow \ [U|\mathbf{0}] = \left[\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{c} x_1 + x_2 - 3x_3 + x_4 = 0 \\ x_2 \overset{f.v.}{=} x_2 \\ x_3 = 0 \\ x_4 \overset{f.v.}{=} x_4 \end{array} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4$$

General solution Finally, by the Nonhomogeneous Principle, the general solution is

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} t \ \text{ for any } s, t \in \mathbb{R}.$$

(c)(pts) Note, using row operations, we have

$$[A|\mathbf{b}] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ -1 & -2 & 2 & 1 & 3 \\ -1 & -2 & 7 & 6 & 2k \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 5 & 5 & 3 \\ 0 & 0 & 10 & 10 & 2k \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 5 & 5 & 3 \\ 0 & 0 & 0 & 0 & -6+2k \end{array}\right]$$

Copyright APPM Do not post

So:

(i) the system will be **inconsistent** if $2k-6 \neq 0 \Rightarrow k \neq 3$,

(ii) the system will have **infinitely many solutions** if $2k-6=0 \Rightarrow k=3$,

(iii) this system will **never have a unique solution** since there will always be at least one free variable.

4. (17 pts)

Consider matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

(a) (5 pts) Calculate the determinant of A; based on your result, state whether A^{-1} exists and why.

(b) (12 pts) Find A^{-1} using Gauss–Jordan elimination procedure (i.e., using augmented matrices).

Solution:

(a) The inverse A^{-1} exists if and only if A is a nonsingular matrix, i.e. $\det[A] \neq 0$.

Expanding the determinant along the first row we get

$$\det[A] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 1\begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 1(2) - 1(1) + 1(0) = 1 \neq 0.$$

Since $\det[A] = 1 \neq 0$, the matrix A is non-singular, and therefore its invertible.

(b) The Gauss–Jordan elimination procedure consists of augmenting the matrix A with the identity matrix I and row-reducing it to $[\,A \mid I\,] \longrightarrow [\,I \mid A^{-1}\,]$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{R_2^*=-R_1+R_2 \\ R_3^*=-R_1+R_3}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_1^*=-R_2+R_1 \\ R_3^*=-R_2+R_3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow{R_2^*=-R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

Therefore,

$$A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

5. (12 pts) For each question below, given the vector space $\mathbb{V}$, determine if the described subset $\mathbb{W}$ is a subspace of $\mathbb{V}$. Explain your answer.

(a) $\mathbb{V} = \mathbb{M}_{33}$. Let $\mathbb{W}$ be the set of 3×3 matrices $[a_{ij}]$ where $a_{ij} \leq 0$ for $i \geq j$.

(b) $\mathbb{V} = \mathbb{P}_2$. Let $\mathbb{W}$ be the set defined by $p(x) = 1 + c_1 x + c_2 x^2$, where $c_1, c_2 \in \mathbb{R}$.

(c) $\mathbb{V} = \mathbb{R}^3$. Let $\mathbb{W}$ be the set defined by $ax_1 + bx_2 + cx_3 = 0$, where $a, b, c \in \mathbb{R}$.

Solution:

(a) This is not a subspace. Pick $c < 0$ so that $[ca_{ij}] > 0$ for $i \geq j$. Thus, $c[a_{ij}] \notin \mathbb{W}$ for $c < 0$.

(b) This is not a subspace. For $\alpha \in \mathbb{R}$, $\alpha p(x) = \alpha + \alpha c_1 x + \alpha c_2 x^2 \notin \mathbb{W}$ for any choice of c_1 or c_2. Alternatively, $\mathbf{0} \notin \mathbb{W}$.

(c) This is a subspace. For any choice of **fixed** a, b, c, $ax_1 + bx_2 + cx_3 = 0$ is a plane through the origin and thus a subspace of $\mathbb{R}^3$. Alternatively, for any a, b, c, the subspace is just $\mathbb{R}^3$.

Copyright APPM Do not post

6. (25 pts) Let $\mathbb{P}_3$ denote the vector space of polynomials of degree at most 3, and define

$$\mathbb{W} = \Big\{ p(t) \in P_3 \;\Big|\; p(1) = 0 \;\text{ and }\; p'(0) = 0 \Big\}.$$

You may assume without proof that $\mathbb{W}$ is a subspace of $\mathbb{P}_3$. *The three parts below are independent of one another and may be done in any order.*

(a) (6 pts) Determine whether each of $q(t) = 2t^3 - 3t^2 + 1$ and $r(t) = t^2 - t$ belongs to $\mathbb{W}$. Justify your answers.

(b) (11 pts) Find a basis for $\mathbb{W}$ and state $\dim \mathbb{W}$. Justify that your set is a basis.

(c) (8 pts) Determine whether $S = \{t^2 - 1,\ t^3 - 1,\ 1,\ t\}$ is a basis for $\mathbb{P}_3$. Justify your answer.

Solution: (a) A polynomial lies in W exactly when it satisfies *both* conditions.

$$q(1) = 2 - 3 + 1 = 0 \checkmark, \qquad q'(t) = 6t^2 - 6t \implies q'(0) = 0 \checkmark$$

so $q \in W$. For r,

$$r(1) = 1 - 1 = 0 \checkmark, \qquad r'(t) = 2t - 1 \implies r'(0) = -1 \neq 0 \times$$

so $r \notin W$: it satisfies the first condition but fails the second.

(b) Write $p(t) = a + bt + ct^2 + dt^3$. The two conditions give

$$p(1) = a + b + c + d = 0, \qquad p'(t) = b + 2ct + 3dt^2 \implies p'(0) = b = 0.$$

Hence $b = 0$ and $a = -c - d$, so every element of W has the form

$$p(t) = (-c - d) + ct^2 + dt^3 = c\left(t^2 - 1\right) + d\left(t^3 - 1\right), \qquad c, d \in \mathbb{R},$$

and therefore $W = \operatorname{span}\{t^2 - 1,\ t^3 - 1\}$. These two polynomials are linearly independent: if $c\left(t^2 - 1\right) + d\left(t^3 - 1\right) = 0$ for all t, comparing the coefficients of t^3 and t^2 gives $d = 0$ and $c = 0$. Being a spanning, linearly independent set, $\{t^2 - 1,\ t^3 - 1\}$ is a basis and $\dim W = \boxed{2}$.

(c) Since $\dim(\mathbb{P}_3) = 4$ and S contains four polynomials, S is a basis if and only if it is linearly independent. Writing each element in coordinates relative to the standard basis $\{1, t, t^2, t^3\}$ and forming the matrix of coordinate rows,

$$\begin{vmatrix} -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = 1 \neq 0,$$

so the four polynomials are linearly independent. Four linearly independent vectors in the four-dimensional space $\mathbb{P}_3$ necessarily form a basis, so S *is* a basis for $\mathbb{P}_3$.

Copyright APPM Do not post