- This exam is worth 100 points and has 6 problems. Do all problems.
- Show all work. Simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOU HAVE SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE!** (Failure to do so may result in a penalty or a score of $0/100$ on this exam.)
- You are taking this exam in a proctored and honor code enforced environment. Calculators, cell phones, or other electronic devices or the internet are not permitted during the exam. One 8.5"$\times$11" crib sheet with writing on one side is allowed.

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0. At the top of the page containing your solution to problem 1, write the following statement: "I will abide by the CU Boulder Honor Code on this exam" and sign your name to it.

1. (10 pts) Write the word TRUE or FALSE as appropriate. *No work need be shown.* No partial credit given. Do not use the abbreviation "T" or "F", completely write out either "TRUE" or "FALSE" depending on your selection. (If any of your answers are hard to locate or difficult to read, points will be deducted.)

   (a) If A and B are invertible $n \times n$ matrices, then their sum $C = A + B$ must be invertible as well.

   (b) Let $A \in \mathbb{M}_{44}$. Then $|3A| = 3^4|A|$.

   (c) A set of vectors is linearly dependent if the zero vector belongs to their span.

   (d) If matrix A has all zeros on its main diagonal, then it is singular.

   (e) The vectors $\mathbf{v}_1 = \left[\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right]$, $\mathbf{v}_2 = \left[\begin{smallmatrix} -1 \\ -5 \end{smallmatrix}\right]$, $\mathbf{v}_3 = \left[\begin{smallmatrix} -3 \\ -6 \end{smallmatrix}\right]$ span $\mathbb{R}^2$.

2. (15 pts)

   Given $A = \begin{bmatrix} 1 & 1 & -2 \\ 3 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, and $\vec{c} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, compute the following expressions or solve for $\vec{x}$.

   (a) AB

   (b) $(BA)^T(B^{-1})^T$

   (c) $(A + B)^{-1}\vec{x} = \vec{c}$

3. (21 pts) The following problems are **not** related. *Do all problems.*

   (a) (6 pts) Use *row operations* to find the **reduced row echelon form** of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 6 \\ 5 & 8 & 12 \\ 0 & 8 & 14 \end{bmatrix}$. Show all steps.

   (b) (8 pts) Suppose A is a nonzero 3×4 matrix and $\mathbf{b}$ satisfies $A\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 2\mathbf{b}$. If $U = \begin{bmatrix} 1 & 1 & -3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is the *row echelon form* of A, find the general solution of $A\mathbf{x} = \mathbf{b}$. Justify your answer.

   (c) (7 pts) For what values of k does the augmented system $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ -1 & -2 & 2 & 1 & 3 \\ -1 & -2 & 7 & 6 & 2k \end{array}\right]$ have (i) no solution? (ii) infinitely many solutions? (iii) a unique solution? Explain.

PROBLEMS #4 – #6 ON THE OTHER SIDE

4. (17 pts)

Consider matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

(a) (5 pts) Calculate the determinant of A; based on your result, state whether A^{-1} exists and why.

(b) (12 pts) Find A^{-1} using Gauss–Jordan elimination procedure (i.e., using augmented matrices).

5. (12 pts) For each question below, given the vector space $\mathbb{V}$, determine if the described subset $\mathbb{W}$ is a subspace of $\mathbb{V}$. Explain your answer.

(a) $\mathbb{V} = \mathbb{M}_{33}$. Let $\mathbb{W}$ be the set of 3×3 matrices $[a_{ij}]$ where $a_{ij} \leq 0$ for $i \geq j$.

(b) $\mathbb{V} = \mathbb{P}_2$. Let $\mathbb{W}$ be the set defined by $p(x) = 1 + c_1 x + c_2 x^2$, where $c_1, c_2 \in \mathbb{R}$.

(c) $\mathbb{V} = \mathbb{R}^3$. Let $\mathbb{W}$ be the set defined by $ax_1 + bx_2 + cx_3 = 0$, where $a, b, c \in \mathbb{R}$.

6. (25 pts) Let $\mathbb{P}_3$ denote the vector space of polynomials of degree at most 3, and define

$$\mathbb{W} = \Big\{ p(t) \in P_3 \;\Big|\; p(1) = 0 \;\text{ and }\; p'(0) = 0 \Big\}.$$

You may assume without proof that $\mathbb{W}$ is a subspace of $\mathbb{P}_3$. *The three parts below are independent of one another and may be done in any order.*

(a) (6 pts) Determine whether each of $q(t) = 2t^3 - 3t^2 + 1$ and $r(t) = t^2 - t$ belongs to $\mathbb{W}$. Justify your answers.

(b) (11 pts) Find a basis for $\mathbb{W}$ and state $\dim \mathbb{W}$. Justify that your set is a basis.

(c) (8 pts) Determine whether $S = \{t^2 - 1,\ t^3 - 1,\ 1,\ t\}$ is a basis for $\mathbb{P}_3$. Justify your answer.