- This exam is worth 100 points and has 5 problems. Do all problems.
- Show all work. Simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOU HAVE SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE!** (Failure to do so may result in a penalty or a score of $0/100$ on this exam.)
- You are taking this exam in a proctored and honor code enforced environment. Calculators, cell phones, or other electronic devices or the internet are not permitted during the exam. One 8.5"×11" crib sheet with writing on one side is allowed.

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0. At the top of the page containing your solution to problem 1, write the following statement: "I will abide by the CU Boulder Honor Code on this exam" and sign your name to it.

1. (22pts) Do all the following problems and show all work.

   (a) (5pts) Find all equilibrium solutions of the ODE $\dfrac{dy}{dt} = t^2y + y + t^2 + 1$.

   (b) (5pts) Classify the stability of the equilibrium solutions of $\dfrac{dy}{dt} = t^2y + y + t^2 + 1$. Explain.

   (c) (12pts) Use **separation of variables** to find the general solution of $\dfrac{dy}{dt} = t^2y + y + t^2 + 1$. Show all work.

   Solution:

   (a)(5pts) Note that $f(t, y) = t^2y+y+t^2+1$ has no domain restrictions and $t^2y+y+t^2+1 = (t^2+1)y+t^2+1 = (t^2+1)(y+1)$ so $\frac{dy}{dt} = 0$ implies $(t^2 + 1)(y + 1) = 0$ so we see that $\boxed{y = -1 \text{ is an equilibrium solution.}}$

   (b)(5pts) Since $\frac{dy}{dt} = (t^2 + 1)(y + 1)$, we have $\frac{dy}{dt} > 0$ when $y > -1$ and, if $y < -1$, then $\frac{dy}{dt} < 0$ so $y = -1$ is an unstable equilibrium solution.

   (c)(12pts) Using separation of variables gives

   $$\frac{dy}{dt} = (t^2 + 1)(y + 1) \;\Rightarrow\; \frac{1}{y+1}dy = (t^2 + 1)dt \;\Rightarrow\; \ln|y + 1| = \frac{t^3}{3} + t + \widetilde{C} \;\Rightarrow\; y + 1 = \pm e^{\widetilde{C}} e^{\frac{t^3}{3}+t}$$

   so $y(t) = Ce^{\frac{t^3}{3}+t} - 1$ if $y \neq -1$. Finally, since $y = -1$ is a solution, it is possible that $C = 0$, thus, the general solution is $\boxed{y(t) = Ce^{\frac{t^3}{3}+t} - 1 \text{ where } C \in \mathbb{R}.}$

2. (24pts) Initially, a tank contains 600 liters of water and 120 kg of sucrose. Water with a sucrose concentration of 0.1 kg/L is added at a rate of 40 L/min. The water mixes instantaneously and exits the tank at a rate of 20 L/min.

   (a) (12pts) Let $x(t)$ be the quantity of sucrose in the tank at time t in minutes. Set up a mixing-model initial value problem that describes the rate of change of the amount of sucrose in the tank at any time t.

   (b) (12pts) Use the **integrating factor method** to solve for $x(t)$. Be sure to show all the steps of the integrating factor method.

   Solution:

   (a)(12pts) If $x(t)$ is the quantity of sucrose in the tank at time t in minutes, then $x(0) = 120$ kg. Note that

   $$\text{Rate In} = \text{Concentration in} \cdot \text{Flow rate in} = 0.1\ \frac{\text{kg}}{\text{L}} \cdot 40\ \frac{\text{L}}{\text{min}} = 4\ \frac{\text{kg}}{\text{min}}.$$

   Since water flows in at 40 L/min and out at 20 L/min, there is a net *inflow* of 20 L per minute, so the tank contains $600 + 20t$ liters at time t, thus

   $$\text{Rate Out} = \text{Concentration out} \cdot \text{Flow rate out} = \frac{x(t)}{600 + 20t}\ \frac{\text{kg}}{\text{L}} \cdot 20\ \frac{\text{L}}{\text{min}} = \frac{x(t)}{t + 30}\ \frac{\text{kg}}{\text{min}}$$

thus, we have the mixing model

$$\frac{dx}{dt} = \text{Rate in} - \text{Rate out} \ \Rightarrow \ \boxed{\frac{dx}{dt} = 4 - \frac{x}{t+30},\ x(0) = 120.}$$

(b)(12pts) We can rewrite the differential equation as $x'(t) + \frac{1}{t+30}x = 4 \Leftrightarrow x'(t) + p(t)x(t) = f(t)$ so the integrating factor is $\mu(t) = e^{\int p(t)dt} = e^{\int \frac{1}{t+30}dt} = e^{\ln(t+30)} = t + 30$ and, therefore, by multiplying both sides of the original equation by the integrating factor we get

$$x'(t) + p(t)x(t) = f(t) \ \Rightarrow \ [\mu(t)x(t)]' = f(t)\mu(t) \ \Rightarrow \ x(t) = \frac{\int f(t)\mu(t)\,dt + C}{\mu(t)} = \frac{\int 4(t+30)\,dt + C}{t+30} = \frac{2t^2 + 120t + C}{t+30}$$

where $120 = x(0) = \frac{C}{30}$ so $C = 3600$, therefore $\boxed{x(t) = \frac{2t^2 + 120t + 3600}{t+30}.}$ If one wishes to simplify the answer further, we could use the fact that $2t^2 + 120t + 3600 = 2(t+30)^2 + 1800$, so $x(t) = 2t + 60 + \frac{1800}{t+30}$.

3. (20pts) Write the word TRUE or FALSE as appropriate. *No work need be shown.* No partial credit given. Do not use the abbreviation "T" or "F", completely write out either "TRUE" or "FALSE" depending on your selection. (If any of your answers are hard to locate or difficult to read, points will be deducted.)

   (a) (4pts) Consider the differential equation $y'(t) = 12 + 16t - 4y$. If we start with $y(0) = 1$ and use one step of Euler's Method to estimate $y(0.5)$, then we get $y(0.5) \approx 5$.

   (b) (4pts) Suppose L is a linear operator. If $L(y_1) = f(t)$ and $L(3y_1 + 4y_2) = 5f(t)$, then $L(y_2) = \frac{f(t)}{2}$.

   (c) (4pts) Picard's Theorem guarantees that the differential equation $y' = ty^{1/3}$, $y(0) = 0$ has a unique solution.

   (d) (4pts) If $y_1(t)$ and $y_2(t)$ are solutions of $y'' - y' = ty$, then the Nonhomogeneous Principle for linear equations says that $y(t) = c_1y_1(t) + c_2y_2(t)$ is also a solution for any constants c_1 and c_2.

   (e) (4pts) The autonomous system $\begin{cases} x'(t) = 1 - x - y, \\ y'(t) = 1 - x^2 - y^2 \end{cases}$ has the line $y = 1 - x$ as its h-nullcline and the circle $x^2 + y^2 = 1$ as its v-nullcline.

Solution:

(a)(4pts) $\boxed{\text{TRUE.}}$ Euler's Method is $y_{n+1} = y_n + hf(t_n, y_n)$ for $n = 0, 1, \ldots$ and, starting with $t_0 = 0$, to estimate $y(0.5)$ in one step, we need $t_1 = 0.5$ so let $h = 0.5$ (since $t_1 = t_0 + h$). Now $(t_0, y_0) = (0, 1)$ and $f(t_n, y_n) = 12 + 16t_n - 4y_n$ implies

$$y(\tfrac{1}{2}) \approx y_1 = y_0 + \frac{1}{2}(12 + 16t_0 - 4y_0) = 1 + \frac{8}{2} = 5.$$

(b)(4pts) $\boxed{\text{TRUE.}}$ Note that $5f(t) = L(3y_1 + 4y_2) = 3L(y_1) + 4L(y_2)$ so $L(y_2) = \frac{5f(t) - 3L(y_1)}{4} = \frac{2f(t)}{4}$.

(c)(4pts) $\boxed{\text{FALSE.}}$ Here $f(t, y) = ty^{1/3}$ is continuous in a region R containing the point $(t_0, y_0) = (0, 0)$ but the derivative with respect to y is $f_y(t, y) = \frac{t}{3y^{2/3}}$ which is only continuous if $y \neq 0$ so Picard's Theorem does not provide any conclusion about uniqueness, that is, in this case we cannot use Picard's Theorem conclude that there is a unique solution or that there is not a unique solution since the converse of Picard's Theorem does not necessarily hold.

(d)(4pts) $\boxed{\text{FALSE.}}$ This result regarding homogeneous linear equations is due to the Superposition Principle and not the Nonhomogeneous Principle. (The Nonhomogeneous Principle states that, for a linear operator L, if $L(y_p) = f(t)$ and $L(y_h) = 0$ then $L(y_p + y_h) = f(t)$.)

(e)(4pts) $\boxed{\text{FALSE.}}$ Recall that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ so $x^2 + y^2 = 1 \Leftrightarrow \frac{dy}{dt} = 0$, thus we have a h-nullcline and $y = 1 - x \Leftrightarrow \frac{dx}{dt} = 0$, so we have a v-nullcline.

4. (18pts) The following problems are not related.

   (a) (6pts) Suppose coffee has a temperature of $200°\text{F}$ when freshly poured, and one minute later has cooled to $190°\text{F}$ in a room at $70°\text{F}$. Assuming the rate of change of the coffee's temperature follows Newton's law of cooling with constant of proportionality k, use Euler's method to estimate the value of k in units of min^{-1}.

(b) (12pts) Suppose the homogeneous solution of $y'(t) = k\big(65 - y(t)\big)$, $y(10) = 35$ is $y_h(t) = Ce^{-kt}$, where $k = \frac{\ln(2)}{10}$ and $C \in \mathbb{R}$. Show all the steps required to find the unique solution $y(t)$ of this initial value problem using the *Euler-Lagrange Two Stage Method* (that is, using **variation of parameters**).

Solution:

(a) (6pts) Newton's law of cooling says $dT_{\text{coffee}}/dt = k(T_{\text{coffee}} - T_{\text{air}})$. Euler's method (forward Euler), then, says

$$\frac{\Delta T_{\text{coffee}}}{\Delta t} \approx k(T_{\text{coffee}} - T_{\text{air}}) \quad \Rightarrow \quad 10^\circ \frac{\text{F}}{1\text{min}} \approx k(200^\circ\text{F} - 70^\circ\text{F}) \quad \Rightarrow \quad \boxed{k \approx \frac{10/130}{\text{min}}} \approx 0.077/\text{min}$$

(b) (12pts) We have $y' + ky = 65k$ where $k = \frac{\ln(2)}{10}$. If the homogeneous solution is $y_h = Ce^{-kt}$, then we start by assuming the particular solution has the form $y_p = v(t)e^{-kt}$, where $v(t)$ is the varying parameter, then

$$\begin{aligned} y_p' + ky_p = 65k \;&\Rightarrow\; [v(t)e^{-kt}]' + kv(t)e^{-kt} = 65k \\ &\Rightarrow\; [v'(t)e^{-kt} - kv(t)e^{-kt}] + kv(t)e^{-kt} = 65k \;\Rightarrow\; v'(t)e^{-kt} = 65k \;\Rightarrow\; v'(t) = 65ke^{kt} \end{aligned}$$

so $v(t) = 65e^{kt}$, thus, the general solution is $y(t) = y_p + y_h = 65 + Ce^{-kt}$. Now, since $10k = \ln(2)$, we have

$$35 = y(10) = 65 + Ce^{-10k} \;\Rightarrow\; Ce^{-\ln(2)} = -30 \;\Rightarrow\; C = -30e^{\ln(2)} \;\Rightarrow\; \boxed{y(t) = 65 - 60e^{-\frac{\ln(2)}{10}t}.}$$

5. (16pts) Consider the following first order ordinary differential equations:

$$(1)\ y' = y^2 - 1, \qquad (2)\ y' = t + y, \qquad (3)\ y' = (t^2 + y^2)/2, \qquad (4)\ y' = ty, \qquad (5)\ y' = t - y,$$

(a) (8pts) Match the direction field below that best fits the given differential equation above. (Note that there are more equations than direction fields.)

(b) (4pts) For each direction field pictured below, identify the equilibria and give their stability where appropriate.

(c) (4pts) In graphs (C) and (D), if a solution curve passes through the point $(t, y) = (1, -1)$ respectively, describe the behaviors of these solution as $t \to +\infty$ respectively.

Solution:

(a)(8pt) We have $\begin{cases} (1) - \text{(B)}, \\ (3) - \text{(A)}, \\ (4) - \text{(C)}, \\ (5) - \text{(D)}. \end{cases}$

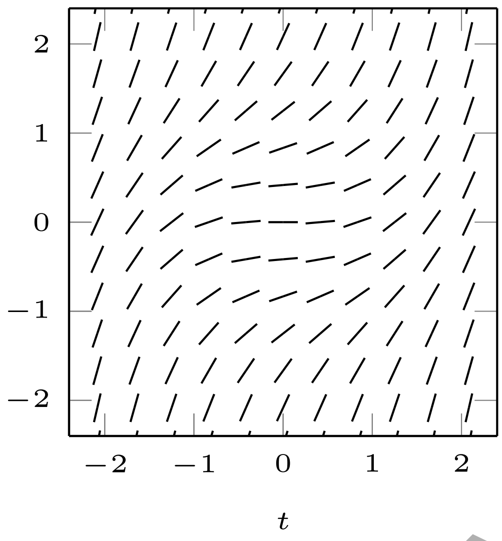
(b)(4pt) For (1)-(B), we have $f(t, y) = y^2 - 1 = (y - 1)(y + 1)$, so $y = -1$ is a stable equilibrium solution and $y = 1$ is an unstable equilibrium solution.

For (4)-(C), we have $f(t, y) = ty$, so $y = 0$ is a *unstable* equilibrium solution. Solutions that start away from $y = 0$ are eventually repelled. *All the other direction fields exhibit no equilibrium solution.*

(c)(4pt) For (C)-(4), if we start a trajectory at $(1, -1)$, then the path is repelled from the equilibrium solution and we have $y \to -\infty$ as $t \to \infty$.

For (D)-(5), if we start a trajectory at $(1, -1)$, then the direction field directs the path to be positively unbounded, that is, $y \to \infty$ as $t \to \infty$.

(A)



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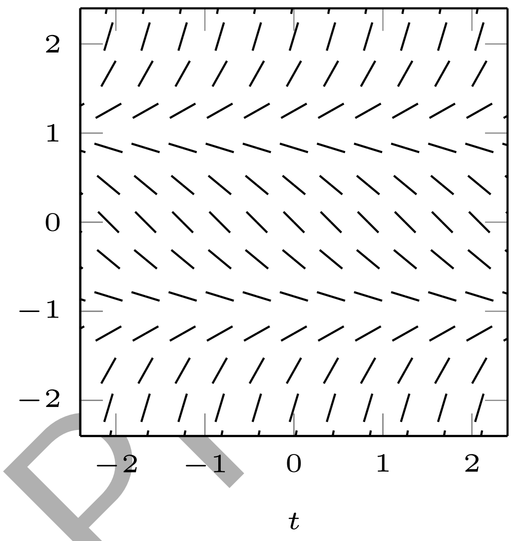
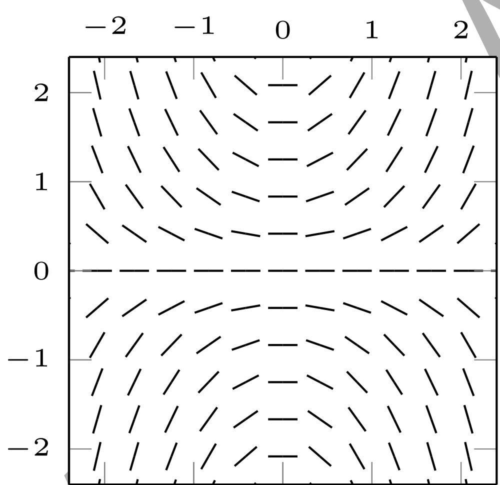
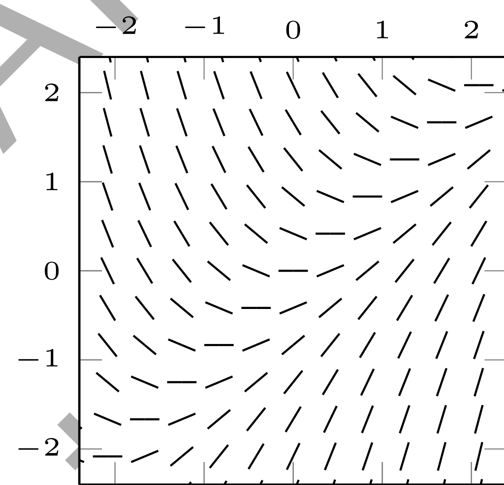




(C)



(D)

END