- This exam is worth 100 points and has 5 problems. Do all problems.
- Show all work. Simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOU HAVE SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE!** (Failure to do so may result in a penalty or a score of $0/100$ on this exam.)
- You are taking this exam in a proctored and honor code enforced environment. Calculators, cell phones, or other electronic devices or the internet are not permitted during the exam. One 8.5"×11" crib sheet with writing on one side is allowed.

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0. At the top of the page containing your solution to problem 1, write the following statement: "I will abide by the CU Boulder Honor Code on this exam" and sign your name to it.

1. (22pts) Do all the following problems and show all work.

   (a) (5pts) Find all equilibrium solutions of the ODE $\dfrac{dy}{dt} = t^2y + y + t^2 + 1$.

   (b) (5pts) Classify the stability of the equilibrium solutions of $\dfrac{dy}{dt} = t^2y + y + t^2 + 1$. Explain.

   (c) (12pts) Use **separation of variables** to find the general solution of $\dfrac{dy}{dt} = t^2y + y + t^2 + 1$. Show all work.

2. (24pts) Initially, a tank contains $600$ liters of water and $120$ kg of sucrose. Water with a sucrose concentration of $0.1$ kg/L is added at a rate of $40$ L/min. The water mixes instantaneously and exits the tank at a rate of $20$ L/min.

   (a) (12pts) Let $x(t)$ be the quantity of sucrose in the tank at time $t$ in minutes. Set up a mixing-model initial value problem that describes the rate of change of the amount of sucrose in the tank at any time $t$.

   (b) (12pts) Use the **integrating factor method** to solve for $x(t)$. Be sure to show all the steps of the integrating factor method.

3. (20pts) Write the word TRUE or FALSE as appropriate. *No work need be shown.* No partial credit given. Do not use the abbreviation "T" or "F", completely write out either "TRUE" or "FALSE" depending on your selection. (If any of your answers are hard to locate or difficult to read, points will be deducted.)

   (a) (4pts) Consider the differential equation $y'(t) = 12 + 16t - 4y$. If we start with $y(0) = 1$ and use one step of Euler's Method to estimate $y(0.5)$, then we get $y(0.5) \approx 5$.

   (b) (4pts) Suppose $L$ is a linear operator. If $L(y_1) = f(t)$ and $L(3y_1 + 4y_2) = 5f(t)$, then $L(y_2) = \dfrac{f(t)}{2}$.

   (c) (4pts) Picard's Theorem guarantees that the differential equation $y' = ty^{1/3}$, $y(0) = 0$ has a unique solution.

   (d) (4pts) If $y_1(t)$ and $y_2(t)$ are solutions of $y'' - y' = ty$, then the Nonhomogeneous Principle for linear equations says that $y(t) = c_1y_1(t) + c_2y_2(t)$ is also a solution for any constants $c_1$ and $c_2$.

   (e) (4pts) The autonomous system $\begin{cases} x'(t) = 1 - x - y, \\ y'(t) = 1 - x^2 - y^2 \end{cases}$ has the line $y = 1 - x$ as its $h$-nullcline and the circle $x^2 + y^2 = 1$ as its $v$-nullcline.

PROBLEMS #4 AND #5 ON THE OTHER SIDE

4. (18pts) The following problems are not related.

   (a) (6pts) Suppose coffee has a temperature of $200°\mathrm{F}$ when freshly poured, and one minute later has cooled to $190°\mathrm{F}$ in a room at $70°\mathrm{F}$. Assuming the rate of change of the coffee's temperature follows Newton's law of cooling with constant of proportionality $k$, use Euler's method to estimate the value of $k$ in units of $\mathrm{min}^{-1}$.

   (b) (12pts) Suppose the homogeneous solution of $y'(t) = k\big(65 - y(t)\big)$, $y(10) = 35$ is $y_h(t) = Ce^{-kt}$, where $k = \frac{\ln(2)}{10}$ and $C \in \mathbb{R}$. Show all the steps required to find the unique solution $y(t)$ of this initial value problem using the *Euler-Lagrange Two Stage Method* (that is, using **variation of parameters**).

5. (16pts) Consider the following first order ordinary differential equations:

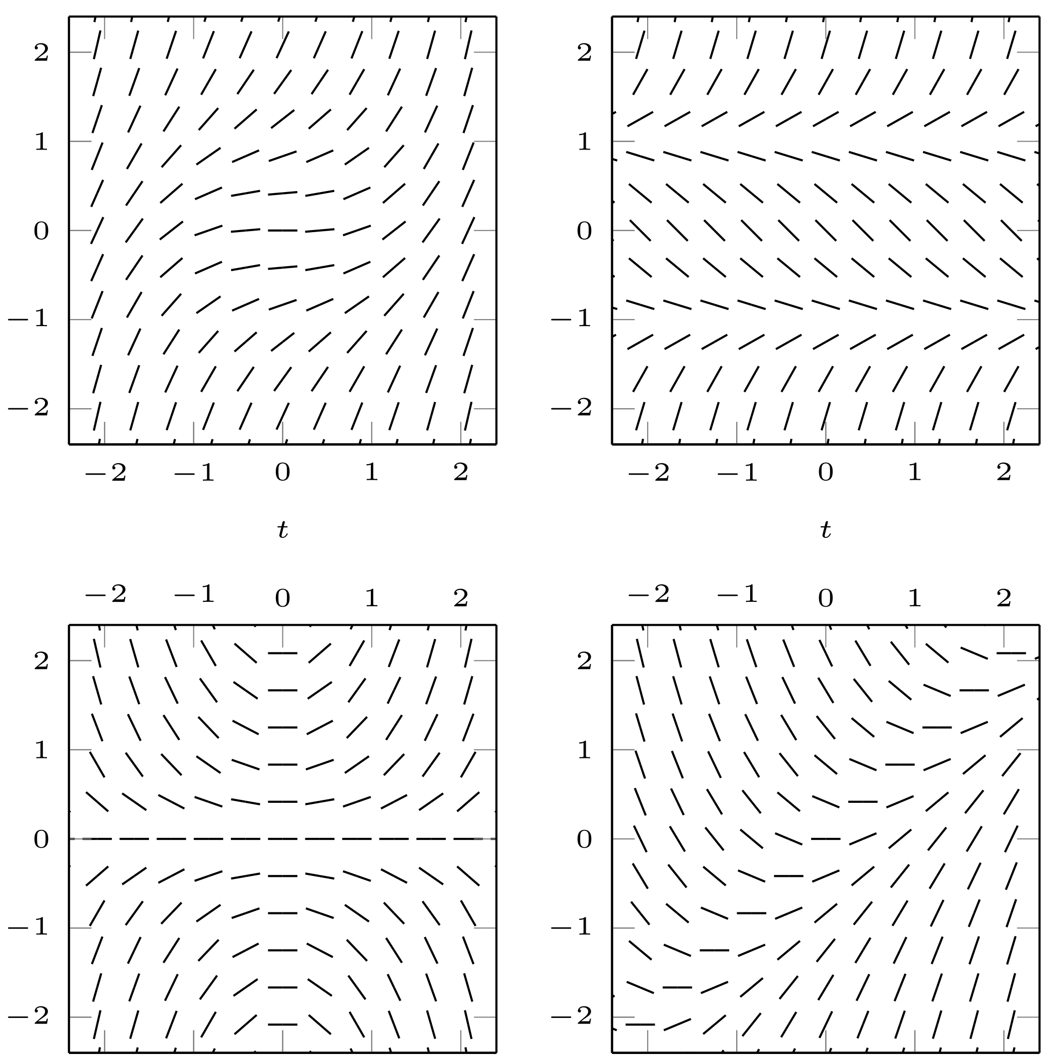
$$(1)\ y' = y^2 - 1, \qquad (2)\ y' = t + y, \qquad (3)\ y' = (t^2 + y^2)/2, \qquad (4)\ y' = ty, \qquad (5)\ y' = t - y,$$

   (a) (8pts) Match the direction field below that best fits the given differential equation above. (Note that there are more equations than direction fields.)

   (b) (4pts) For each direction field pictured below, identify the equilibria and give their stability where appropriate.

   (c) (4pts) In graphs (C) and (D), if a solution curve passes through the point $(t, y) = (1, -1)$ respectively, describe the behaviors of these solution as $t \to +\infty$ respectively.

(A) (B)




(C) (D)

$\boxed{\text{END}}$