Write your name below. This exam has 5 problems and is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You are allowed to use one page of notes. You cannot collaborate on the exam or seek outside help, nor can you use a calculator or any computational software.

Name:

Do not begin until instructed.

1. (15 points) Consider the three vectors:

$$\mathbf{v}_1 = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$$

$$\mathbf{v}_2 = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$$

$$\mathbf{v}_3 = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

(a) (5 points) What is the volume of the parallepiped formed by these three vectors?

(b) (10 points) Calculate the following. If an expression does not exist, indicate this by writing "DNE".

i. $(\mathbf{v}_1 - \mathbf{v}_2) \times \mathbf{v}_3$

ii. $(\mathbf{v}_1 \cdot \mathbf{v}_2) \times \mathbf{v}_3$

iii. $(\mathbf{v}_1 \cdot \mathbf{v}_2)\mathbf{v}_3$

iv. $\mathbf{v}_2 \times (\mathbf{v}_3 \times \mathbf{v}_1) + \mathbf{v}_2 \times (\mathbf{v}_1 \times \mathbf{v}_3)$

Solution

(a) The volume is given by the absolute value of the triple product:

$$V = |\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)|$$

Let's first calculate the cross product using the determinant method:

$$\begin{aligned}\mathbf{v}_2 \times \mathbf{v}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 1 & 1 \end{vmatrix} &= ((-1)(1) - (3)(1))\mathbf{i} + ((3)(1) - (2)(1))\mathbf{j} + ((2)(1) - (-1)(1))\mathbf{k} \\ &= -4\mathbf{i} + \mathbf{j} + 3\mathbf{k}\end{aligned}$$

Taking the absolute value of the dot product with $\mathbf{v}_1$ gives the volume:

$$\begin{aligned}|\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)| &= |(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (-4\mathbf{i} + \mathbf{j} + 3\mathbf{k})| \\ &= |-12 - 1 + 6| \\ &= 7\end{aligned}$$

(b) i. We take the difference $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{i} - \mathbf{k}$ and then take the cross product:

$$\begin{aligned}(\mathbf{v}_1 - \mathbf{v}_2) \times \mathbf{v}_3 &= (\mathbf{i} - \mathbf{k}) \times (\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= \mathbf{i} \times \mathbf{i} + \mathbf{i} \times \mathbf{j} + \mathbf{i} \times \mathbf{k} - \mathbf{k} \times \mathbf{i} - \mathbf{k} \times \mathbf{j} - \mathbf{k} \times \mathbf{k} \\ &= \mathbf{0} + \mathbf{k} - \mathbf{j} - \mathbf{j} + \mathbf{i} + \mathbf{0} \\ &= \mathbf{i} - 2\mathbf{j} + \mathbf{k}\end{aligned}$$

ii. This product **DNE** since $\mathbf{v}_1 \cdot \mathbf{v}_2$ is a scalar. The cross product is only defined between two vectors.

iii. This product does exist.

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_2 &= (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 6 + 1 + 6 = 13 \\ (\mathbf{v}_1 \cdot \mathbf{v}_2)\mathbf{v}_3 &= 13(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= 13\mathbf{i} + 13\mathbf{j} + 13\mathbf{k} \end{aligned}$$

iv. Since $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, we can use this property of cross products to simplify:

$$\begin{aligned} \mathbf{v}_2 \times (\mathbf{v}_3 \times \mathbf{v}_1) + \mathbf{v}_2 \times (\mathbf{v}_1 \times \mathbf{v}_3) &= \mathbf{v}_2 \times (\mathbf{v}_3 \times \mathbf{v}_1) - \mathbf{v}_2 \times (\mathbf{v}_3 \times \mathbf{v}_1) \\ &= \mathbf{0} \end{aligned}$$

2. (20 points) A line is given by the symmetric equations

$$\left(\frac{x-3}{2}\right) = \left(\frac{y-2}{3}\right) = (z-1)$$

(a) (5 points) What is the vector equation for this line?

(b) (5 points) How far is the point $P(1,2,3)$ from the line?

(c) (10 points) Find the point where the line above intersects the line given by

$$\mathbf{r}(t) = \langle 1,2,3\rangle + \langle 1,2,3\rangle\, t$$

If the lines do not intersect, indicate whether they are parallel or skew.

Solution

(a) Setting each term in the symmetric equations equal to the parameter s, we solve to get the parametric equations:

$$\frac{x-3}{2} = s \to x = 2s+3$$

$$\frac{y-2}{3} = s \to y = 3s+2$$

$$z-1 = s \to z = s+1$$

From these we read off each component of the vector equation and find that

$$\mathbf{l}(s) = \langle 3,2,1\rangle + \langle 2,3,1\rangle\, s$$

(b) We first find a point on the line. Let's choose the point at $t=0$ (for simplicity) to get the point $Q(3,2,1)$. Then find the vector from Q to P:

$$\overrightarrow{QP} = \langle 1-3, 2-2, 3-1\rangle = \langle -2,0,2\rangle$$

Now we calculate the distance:

$$d = \frac{1}{|\mathbf{v}|}|\mathbf{v}\times\overrightarrow{QP}|$$

Find the cross product:

$$\begin{aligned}\mathbf{v}\times\overrightarrow{QP} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ -2 & 0 & 2 \end{vmatrix} \\ &= \langle 6,-6,6\rangle\end{aligned}$$

Bringing this together gives

$$d = \frac{|\,\langle 6,-6,6\rangle\,|}{|\,\langle 2,3,1\rangle\,|} = \frac{6\sqrt{3}}{\sqrt{14}}$$

(c) We set the two vector equations for the lines equal to each other and see if there are parameters s and t where the three components are all equal. This creates a system of three equations, one for each component:

$$\begin{aligned} x: &\quad 3+2s = 1+t \\ y: &\quad 2+3s = 2+2t \\ z: &\quad 1+s = 3+3t \end{aligned}$$

We choose two of these equations and solve them for s and t. We then check if the values satisfy the remaining equation. First use the x equation to get t in terms of s:

$$\begin{aligned} 3+2s &= 1+t \\ 2+2s &= t \end{aligned}$$

Now substitute t into the y equation and solve for s:

$$\begin{aligned} 2+3s &= 2+2(2+2s) \\ 3s &= 4+4s \\ s &= -4 \end{aligned}$$

Substitute this s value into the x equation to get t:

$$\begin{aligned} 3+2(-4) &= 1+t \\ -6 &= t \end{aligned}$$

So we have both values now. We check to see if the satisfy the z equation:

$$\begin{aligned} \text{LHS} &= 1+(-4) = -3 \\ \text{RHS} &= 3+3(-6) = -15 \end{aligned}$$

Since these are not equal, the lines do not intersect. They also are not parallel, since they have vectors that aren't multiples of each other. They are therefor skew.

3. (25 points) Consider the plane given by

$$3x - 2y + z = 2$$

and the line given by the symmetric equations

$$\frac{x+1}{2} = y = 2 - z.$$

(a) (5 points) Find the point where the line intersects the plane.

(b) (5 points) Find the angle between the line and plane's normal vector.

(c) (5 points) Show that the point A$(3, 2, 0)$ is on the line.

(d) (10 points) Determine how far the point A is from the plane.

Solution

(a) We first need to parameterize the line. Setting each part of the symmetric equations equal to the parameter t, we solve for the variables to find the parametric equations :

$$\begin{aligned} \frac{x+1}{2} &= t & &\to x = 2t - 1 \\ y &= t & &\to y = t \\ 2 - z &= t & &\to z = 2 - t \end{aligned}$$

We substitute these functions into the equation for the plane and solve for the parameter:

$$\begin{aligned} 3x - 2y + z &= 2 \\ 3(2t - 1) - 2(t) + (2 - t) &= 2 \\ 6t - 3 - 2t + 2 - t &= 2 \\ 3t &= 3 \\ t &= 1 \end{aligned}$$

Substitute this t value into our parametric equations to find our point:

$$\begin{aligned} &x = 2 \cdot 1 - 1 = 1 \\ &y = 1 \\ &z = 2 - 1 = 1 \end{aligned}$$

So the point is $(1, 1, 1)$

(b) The normal vector of the plane can be read off from the equation:

$$\mathbf{n} = \langle 3, -2, 1 \rangle$$

While the vector for the line is given by the coefficients of t in the parametric equations:

$$\mathbf{v} = \langle 2, 1, -1 \rangle$$

We can now calculate the cosine of the angle between these vectors:

$$\begin{aligned}\cos\theta &= \frac{\mathbf{n}\cdot\mathbf{v}}{|\mathbf{n}||\mathbf{v}|}\\ &= \frac{3\cdot 2+(-2)\cdot 1+1\cdot(-1)}{\sqrt{3^2+(-2)^2+1^2}\sqrt{2^2+1^2+(-1)^2}}\\ &= \frac{3}{\sqrt{14}\sqrt{6}}\\ &= \frac{3}{2\sqrt{21}}\end{aligned}$$

So we have

$$\theta = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right)$$

(c) Simply substituting into the symmetric equations gives

$$\begin{aligned}\frac{3+1}{2} &= 2\\ 2 &= 2\\ 2-0 &= 2\end{aligned}$$

All the terms are equal, so the point is on the line.

(d) We already have a point in the plane, so we can find the vector between it and A:

$$\mathbf{v} = \langle 3,2,0\rangle - \langle 1,1,1\rangle = \langle 2,1,-1\rangle$$

The absolute value of the component of this vector along the normal vector gives the distance:

$$\begin{aligned}\text{comp}_{\mathbf{n}}\mathbf{v} &= \frac{\mathbf{n}\cdot\mathbf{v}}{|\mathbf{n}|}\\ &= \frac{\langle 3,-2,1\rangle\cdot\langle 2,1,-1\rangle}{|\langle 3,-2,1\rangle|}\\ &= \frac{6-2-1}{\sqrt{9+4+1}}\\ &= \frac{3}{\sqrt{14}}\end{aligned}$$

4. (25 points) A quadric surface is defined by the equation

$$2x^2 - 4x + y^2 + 4y - z^2 = 0$$

(a) (7 points) Write the equation of the surface in standard form.
(b) (4 points) Classify the surface and state it's orientation. What is the center of the surface?
(c) (7 points) Sketch the trace for $x = 1$. Label all y and z intercepts.
(d) (7 points) Sketch the trace for $z = 6$. Label all x and y intercepts.

Solution

(a) We need to complete the square for x and y:

$$\begin{aligned}
2x^2 - 4x + y^2 + 4y - z^2 &= 0 \\
2(x^2 - 2x) + (y^2 + 4y) - z^2 &= 0 \\
2(x^2 - 2x + 1) + (y^2 + 4y + 4) - z^2 &= 0 + 2 + 4 \\
2(x-1)^2 + (y+2)^2 - z^2 &= 6 \\
\frac{(x-1)^2}{3} + \frac{(y+2)^2}{6} - \frac{z^2}{6} &= 1
\end{aligned}$$

(b) This is a hyperboloid of one sheet opening along z. It's center is at $(1, -2, 0)$

(c) Setting $x = 1$ gives us

$$\frac{(y+2)^2}{6} - \frac{z^2}{6} = 1$$

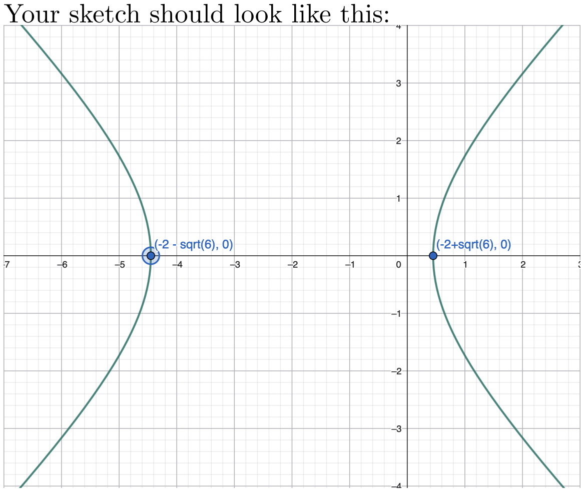
This is a hyperbola centered at $(-2, 0)$. The y intercepts occur where $z = 0$, so we have

$$\begin{aligned}
\frac{(y+2)^2}{6} &= 1 \\
(y+2)^2 &= 6 \\
y + 2 &= \pm\sqrt{6} \\
y &= -2 \pm \sqrt{6}
\end{aligned}$$

There are no z intercepts, since they would occur when $y = 0$ and then we have

$$-\frac{z^2}{6} = 1$$

which has no real solutions.
Your sketch should look like this:



(d) Setting $z = 6$ we get

$$\begin{aligned}
\frac{(x-1)^2}{3} + \frac{(y+2)^2}{6} - \frac{36}{6} &= 1 \\
\frac{(x-1)^2}{3} + \frac{(y+2)^2}{6} - 6 &= 1 \\
\frac{(x-1)^2}{3} + \frac{(y+2)^2}{6} &= 7 \\
\frac{(x-1)^2}{21} + \frac{(y+2)^2}{42} &= 1
\end{aligned}$$

This is an ellipse.

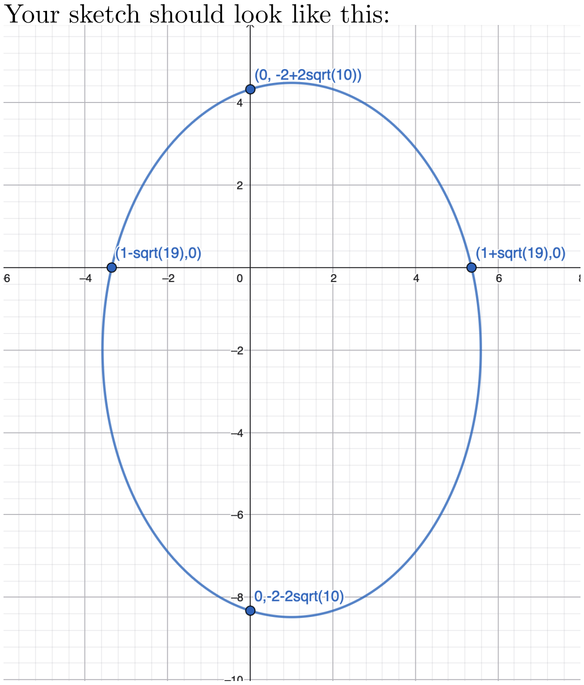
The x intercepts occur where $y = 0$:

$$\begin{aligned}
\frac{(x-1)^2}{21} + \frac{(2)^2}{42} &= 1 \\
\frac{(x-1)^2}{21} &= 1 - \frac{4}{42} \\
\frac{(x-1)^2}{21} &= \frac{38}{42} \\
\frac{(x-1)^2}{21} &= \frac{19}{21} \\
(x-1)^2 &= 19 \\
x &= 1 \pm \sqrt{19}
\end{aligned}$$

The y intercepts occur where $x = 0$:

$$\begin{aligned}
\frac{(-1)^2}{21} + \frac{(y+2)^2}{42} &= 1 \\
\frac{(y+2)^2}{42} &= 1 - \frac{1}{21} \\
\frac{(y+2)^2}{42} &= \frac{20}{21} \\
(y+2)^2 &= 40 \\
y + 2 &= \pm\sqrt{40} \\
y &= -2 \pm 2\sqrt{10}
\end{aligned}$$

Your sketch should look like this:



5. (15 points) An ion is moving in a magnetic field along the space curve given by

$$\mathbf{r}(t) = \frac{1}{\sqrt{2}}\left\langle \cos(t), \sqrt{2}\sin(t), \cos(t)\right\rangle$$

(a) (10 points) Find the velocity vector, $\mathbf{r}'(\mathbf{t})$, and acceleration vector, $\mathbf{r}''(t)$ of the ion. What is their dot product?

(b) (5 points) When $t = \frac{\pi}{4}$, the magnetic field is switched off, causing the ion to move in a straight line. Find an equation for this line.

Solution

(a) We take derivatives of $\mathbf{r}(t)$ to find these vectors:

$$\begin{aligned}\mathbf{v}(t) = \mathbf{r}'(t) &= \frac{1}{\sqrt{2}}\left\langle -\sin(t), \sqrt{2}\cos(t), -\sin(t)\right\rangle \\ \mathbf{a}(t) = \mathbf{r}''(t) &= \frac{1}{\sqrt{2}}\left\langle -\cos(t), -\sqrt{2}\sin(t), -\cos(t)\right\rangle\end{aligned}$$

We now take the dot product:

$$\begin{aligned}\mathbf{v}(t)\cdot\mathbf{a}(t) &= \frac{1}{(\sqrt{2})^2}\Big(\sin(t)\cos(t) - (\sqrt{2})^2\sin(t)\cos(t) + \sin(t)\cos(t)\Big) \\ &= \frac{1}{2}\left(2\sin(t)\cos(t) - 2\sin(t)\cos(t)\right) \\ &= \mathbf{0}\end{aligned}$$

(b) We find the position vector and velocity when $t = \frac{\pi}{4}$:

$$\begin{aligned}\mathbf{r}\left(\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}}\left\langle \frac{1}{\sqrt{2}}, \sqrt{2}\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle \\ &= \left\langle \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right\rangle \\ \mathbf{r}'\left(\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}}\left\langle -\frac{1}{\sqrt{2}}, \sqrt{2}\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\rangle \\ &= \left\langle -\frac{1}{2}, \frac{1}{\sqrt{2}}, -\frac{1}{2}\right\rangle\end{aligned}$$

Choosing our parameter to be s, we have

$$\mathbf{t}(s) = \left\langle \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right\rangle + \left\langle -\frac{1}{2}, \frac{1}{\sqrt{2}}, -\frac{1}{2}\right\rangle s$$