

1. (12 pts) Consider the rational function

$$r(x) = \frac{3x^2 + 7}{(x-2)(x^2-4)(x^2+x+9)^2}.$$

Write out the form of the partial fraction decomposition, but DO NOT solve for the coefficients.

Solution:

$$r(x) = \frac{A}{x+2} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{Dx+E}{x^2+x+9} + \frac{Fx+G}{(x^2+x+9)^2}$$

2. (36 pts) Evaluate the following integrals:

(a) $\int (2 \sin \theta + \cot \theta)^2 d\theta$

(b) $\int x \ln(x+1) dx$

(c) $\int \frac{\sqrt{x^2+4}}{x^4} dx$ *Hint: $\tan \theta$ and $\sec \theta$ can be expressed in terms of $\sin \theta$ and $\cos \theta$.*

Solution:

(a)

$$\begin{aligned} \int (2 \sin \theta + \cot \theta)^2 d\theta &= \int (4 \sin^2 \theta + 4 \cos \theta + \cot^2 \theta) d\theta \\ &= \frac{4}{2} \int (1 - \cos(2\theta)) d\theta + 4 \int \cos \theta d\theta + \int (\csc^2 \theta - 1) d\theta \\ &= 2\theta - \sin 2\theta + 4 \sin \theta - \cot \theta - \theta + C \\ &= \theta - \sin 2\theta + 4 \sin \theta - \cot \theta + C \end{aligned}$$

- (b) Using integration by parts with $u = \ln(x+1)$, $du = \frac{dx}{x+1}$, $dv = x dx$, and $v = \frac{x^2}{2}$, we have

$$\begin{aligned} \int x \ln(x+1) dx &= \frac{x^2}{2} \ln(x+1) - \frac{1}{2} \int \frac{x^2}{x+1} dx \\ &= \frac{x^2}{2} \ln(x+1) - \frac{1}{2} \int \left(x - 1 + \frac{1}{x+1} \right) dx \\ &= \frac{x^2}{2} \ln(x+1) - \frac{1}{2} \left(\frac{x^2}{2} - x + \ln|x+1| \right) + C \end{aligned}$$

- (c) We'll use the trig substitution $x = 2 \tan \theta$, $dx = 2 \sec^2 \theta d\theta$:

$$\begin{aligned} \int \frac{\sqrt{x^2+4}}{x^4} dx &= \int \frac{\sqrt{4 \tan^2 \theta + 4}}{16 \tan^4 \theta} 2 \sec^2 \theta d\theta \\ &= \frac{1}{4} \int \frac{\sqrt{\sec^2 \theta}}{\tan^4 \theta} \sec^2 \theta d\theta \\ &= \frac{1}{4} \int \frac{\sec^3 \theta}{\tan^4 \theta} d\theta \\ &= \frac{1}{4} \int \frac{\cos \theta}{\sin^4 \theta} d\theta \end{aligned}$$

Using the u -substitution $x = \sin \theta$, $du = \cos \theta d\theta$

$$\begin{aligned} &= \frac{1}{4} \int \frac{du}{u^4} \\ &= -\frac{1}{12 \sin^3 \theta} + C \end{aligned}$$

Drawing an appropriate triangle which includes $x = 2 \tan \theta$ shows that $\sin \theta = \frac{\sqrt{x^2 + 4}}{x}$. Hence, we have

$$\int \frac{\sqrt{x^2 + 4}}{x^4} dx = -\frac{1}{12} \left(\frac{\sqrt{x^2 + 4}}{x} \right)^3 + C$$

3. (20 pts) Determine whether the following integrals are convergent or divergent. If the integral converges, find its value; if you use the Comparison Test, state this and fully justify it.

(a) $\int_6^\infty \frac{(1 + \frac{1}{x})}{\sqrt{x^2 - x - 3}} dx$

(b) $\int_0^2 \frac{1-x}{\sqrt{x}} dx$

Solution:

(a)

$$\int_6^\infty \frac{(1 + \frac{1}{x})}{\sqrt{x^2 - x - 3}} dx = \int_6^\infty \frac{1}{\sqrt{x^2 - x - 3}} dx + \int_6^\infty \frac{1}{x\sqrt{x^2 - x - 3}} dx$$

Consider the first integral on the right-hand side of the preceding equation: $\int_6^\infty \frac{1}{\sqrt{x^2 - x - 3}} dx$.

$$0 < x^2 - x - 3 < x^2, \quad [6, \infty)$$

$$0 < \frac{1}{x^2} < \frac{1}{x^2 - x - 3}, \quad [6, \infty)$$

$$0 < \frac{1}{\sqrt{x^2}} < \frac{1}{\sqrt{x^2 - x - 3}}, \quad [6, \infty)$$

$$0 < \frac{1}{x} < \frac{1}{\sqrt{x^2 - x - 3}}, \quad [6, \infty)$$

(Note: The fact that $0 < x^2 - x - 3$ on $[6, \infty)$ can be inferred from its presence inside a square root in the problem statement, and by accepting that the problem is well-posed. With that having been said, it is not difficult to validate that assertion.)

$\int_6^\infty \frac{1}{x} dx$ diverges per the p -test, where $p = 1$.

Therefore, since $\frac{1}{x}$ and $\frac{1}{\sqrt{x^2 - x - 3}}$ are both positive and continuous on $[6, \infty)$ and $\int_6^\infty \frac{1}{x} dx$

diverges, then the Comparison Test indicates that $\int_6^{\infty} \frac{1}{\sqrt{x^2 - x - 3}} dx$ must diverge as well.

It follows that the original integral also diverges

(b)

$$\begin{aligned} \int_0^2 \frac{1-x}{\sqrt{x}} dx &= \lim_{t \rightarrow 0^+} \int_t^2 (x^{-1/2} - x^{1/2}) dx \\ &= \lim_{t \rightarrow 0^+} \left(2x^{1/2} - \frac{2}{3} x^{3/2} \right) \Big|_t^2 \\ &= \lim_{t \rightarrow 0^+} \left(2\sqrt{2} - \frac{2}{3} \cdot (2\sqrt{2}) - 2\sqrt{t} + \frac{2}{3} t^{3/2} \right) \\ &= \frac{6\sqrt{2} - 4\sqrt{2}}{3} = \boxed{\frac{2\sqrt{2}}{3}} \end{aligned}$$

Since the limit is a finite value, the original integral converges

4. (16 pts) Consider the table below, which provides values of $f(x)$ and some of its derivatives. You may also assume that $f''(x)$ is decreasing for all $x \geq 0$.

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$f(x)$	3	$\frac{1}{2}$	$-\frac{1}{9}$	$-\frac{3}{7}$	$-\frac{2}{3}$	$-\frac{3}{4}$	$-\frac{4}{5}$
$f''(x)$	$\frac{3}{5}$	$\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{5}$	$-\frac{12}{5}$	-3	$-\frac{7}{2}$

- (a) Write out an approximation for $\int_1^3 f(x) dx$ using the Trapezoidal Rule with $n = 4$. *You DO NOT need to simplify your answer, but it should be an expression which is easy to enter into a calculator.*
- (b) Give a range of n -values where the trapezoidal rule would give an approximation of $\int_1^3 f(x) dx$ with an error less than or equal to 10^{-3} . *Again, you do not need to simplify your answer, but it should be an expression which is easy to enter into a calculator.*

Solution:

- (a) The region of integration is $x \in [a = 1, b = 3]$. Dividing the region into $n = 4$ subdivisions with spacing

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2} = 0.5$$

gives the grid points

$$x_0 = 1.0, x_1 = 1.5, x_2 = 2.0, x_3 = 2.5, x_4 = 3.0.$$

At these grid points, the function values can be read off from the provided table, giving:

$$f(x_0) = -1/9, f(x_1) = -3/7, f(x_2) = -2/3, f(x_3) = -3/4, f(x_4) = -4/5.$$

The trapezoidal rule is:

$$\int_1^3 f(x)dx \approx \Delta x \left(\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + f(x_3) + \frac{1}{2}f(x_4) \right)$$

Substituting the values into the trapezoidal rule gives:

$$\begin{aligned} \int_1^3 f(x)dx &\approx \frac{1}{2} \left(-\frac{1}{2} \frac{1}{9} - \frac{3}{7} - \frac{2}{3} - \frac{3}{4} - \frac{1}{2} \frac{4}{5} \right), \\ &= \boxed{-\frac{1}{2} \left(\frac{1}{18} + \frac{3}{7} + \frac{2}{3} + \frac{3}{4} + \frac{4}{10} \right)}, \\ &= -\frac{2899}{2520} \quad (\text{if simplified}). \end{aligned}$$

(b) The bound for the approximation error for the trapezoidal rule is

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}$$

where K is an upper bound for $|f''(x)|$.

For the error to be less than 10^{-3} , we must have

$$\frac{K(b-a)^3}{12n^2} < 10^{-3}.$$

For the given integral, $b = 3$ and $a = 1$. The maximum of $|f''(x)|$ between $1 \leq x \leq 3$ can be inferred from the table; this happens at $x = 3$, at which $|f''(x)| = |-7/2| = 7/2$. Hence, $|f''(x)| \leq \frac{7}{2}$ on the interval $[1, 3]$.

Plugging in,

$$\frac{(7/2)2^3}{12n^2} < 10^{-3} \implies n^2 > \frac{2^3 \cdot 7 \cdot 1000}{2 \cdot 12} \implies \boxed{N > \sqrt{\frac{7000}{3}}}.$$

5. (16 pts) Determine if the following statements are **True** or **False**, and give a short justification for your answer. If you believe (a) or (b) are false, give a specific example which shows that the statement is false.

(a) If $\int_{-\infty}^{\infty} f(x) dx$ converges, then both $\int_{-\infty}^1 f(x) dx$ and $\int_1^{\infty} f(x) dx$ must converge as well.

(b) Suppose $a(x)$ and $b(x)$ are positive functions with $a(x) \leq b(x)$ on $[1, \infty)$. If $\int_1^{\infty} a(x) dx$ converges, then $\int_1^{\infty} b(x) dx$ does as well.

(c) The function $f(x) = \frac{2x^2 + \sqrt{x} - 1}{(x+2)(x-1)(x^2+4)}$ has a decomposition given by

$$\frac{A}{x+2} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4}.$$

(d) Suppose $x = 3 \sec \theta$. Then $\sin \theta \tan \theta = \frac{x^2 - 9}{3x}$.

Solution:

- (a) True. This follows from the definition of integration on the interval $(-\infty, \infty)$.
- (b) False. Note that for $a(x) = \frac{1}{x^2}$ and $b(x) = \frac{1}{x}$, we have that $a(x) \leq b(x)$ on $[1, \infty)$ and $\int_1^\infty a(x) dx$ converges, but $\int_1^\infty b(x) dx$ diverges (both by the p -test).
- (c) False. The function $f(x)$ is not rational, since the numerator is not a polynomial; hence, it can't have that partial fraction decomposition.
- (d) True. Drawing a triangle with $x = 3 \sec \theta$ (so the side adjacent to θ is 3, and the hypotenuse is x) gives

$$\sin \theta = \frac{\sqrt{x^2 - 9}}{x}$$
$$\tan \theta = \frac{\sqrt{x^2 - 9}}{3}$$

Multiplying these two together gives the desired result.