1. (25 pts) Parts (a) - (d) are not related to each other.

   (a) i. Identify all value(s) of $x$, if any, for which the rational expression $\dfrac{x^2+7x+10}{x^2+2x-15}$ is undefined.

   ii. Fully simplify the expression in part (i).

   **Solution:**

   i. The given rational function is undefined at values of $x$ for which the polynomial in the denominator equals zero. The factored form of that polynomial is $x^2+2x-15=(x-3)(x+5)$, which equals zero at $\boxed{x=-5,3}$

   ii. Expressing both the numerator and denominator polynomials in factored form leads to the following simplification:

$$\frac{x^2+7x+10}{x^2+2x-15}=\frac{(x+2)(x+5)}{(x-3)(x+5)}=\boxed{\frac{x+2}{x-3}}$$

   (b) Find all solutions, if any, of the following equations.

   i. $x^{-1}(x^2-3x+2)=0$

   ii. $x^{5/2}=9x^{1/2}$

   **Solution:**

   i.

$$x^{-1}(x^2-3x+2)=0$$
$$x^{-1}(x-1)(x-2)=0$$

   Any value of $x$ for which any of the preceding three factors equals zero is a solution to the original equation.

$$x-1=0 \ \Rightarrow \ x=1$$
$$x-2=0 \ \Rightarrow \ x=2$$

   Note that $x^{-1}=1/x$ does not equal zero for any value of $x$.

   Therefore, the set of solutions is $\boxed{x=1,2}$

ii.

$$x^{5/2} = 9x^{1/2}$$

$$x^{5/2} - 9x^{1/2} = 0$$

$$x^{1/2}(x^2 - 9) = 0$$

$$x^{1/2}(x-3)(x+3) = 0$$

Any value of $x$ for which any of the preceding three factors equals zero is a solution to the original equation.

$$x^{1/2} = 0 \;\Rightarrow\; x = 0$$

$$x - 3 = 0 \;\Rightarrow\; x = 3$$

$$x + 3 = 0 \;\Rightarrow\; x = -3$$

Therefore, the set of solutions is $\boxed{x = -3, 0, 3}$

(c) Fully simplify the following complex fraction by expressing it as a rational expression (a quotient of two polynomials):

$$\frac{\frac{x+2}{7}}{\frac{1}{x^2}+\frac{1}{x-1}}$$

You may express the polynomials in either factored form or expanded form.

**Solution:**

$$\begin{aligned}\frac{\frac{x+2}{7}}{\frac{1}{x^2}+\frac{1}{x-1}} &= \frac{\frac{x+2}{7}}{\left(\frac{(x-1)+x^2}{x^2(x-1)}\right)} \\ &= \frac{x+2}{7}\cdot\frac{x^2(x-1)}{(x-1)+x^2} \\ &= \boxed{\frac{x^2(x-1)(x+2)}{7(x^2+x-1)}}\end{aligned}$$

(d) Solve the inequality $-5 < 2 - x \leq -3$. Express your answer using interval notation.

**Solution:**

$$-5 < 2 - x \leq -3$$

$$-7 < -x \leq -5$$

$$7 > -x \geq 5 \qquad \text{(the directions of the inequalities reverse when multiplying by a negative number)}$$

Therefore, the answer in interval notation is $\boxed{[5,7)}$

2. (25 pts) Parts (a) and (b) are not related to each other.

   (a) For parts i-iii, let point A be $(-6, 1)$, let point B be $(3, -3)$, let segment AB be the line segment connecting points A and B, and let point M be the midpoint of segment AB.

      i. Find the $(x, y)$ coordinates of point M.

      **Solution:**

      The general expression for the midpoint of the line segment connecting the points $(x_1, y_1)$ and $(x_2, y_2)$ is

      $$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

      Substituting the coordinate values of points A and B leads to the following midpoint of segment AB:

      $$\left(\frac{-6 + 3}{2}, \frac{1 - 3}{2}\right) = \boxed{\left(-\frac{3}{2}, -1\right)}$$

      ii. Find the length of segment AB.

      **Solution:**

      The length of segment AB is the distance between points A and B, so we'll apply the distance formula.

      $$\begin{aligned} D &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3 - (-6))^2 + (-3 - 1)^2} \\ &= \sqrt{9^2 + (-4)^2} \\ &= \boxed{\sqrt{97}} \end{aligned}$$

iii. Find an equation of the line that is perpendicular to segment AB and passes through point B.

**Solution:**

The first step is to determine the slope of segment AB. We'll let $m_1$ denote that slope.

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - 1}{3 - (-6)} = -\frac{4}{9}$$

The slopes of two perpendicular lines are negative reciprocals of each other. Since the slope of segment AB is $-4/9$, the slope of a line that is perpendicular to segment AB is $9/4$.

The point-slope form of the equation of the line passing through a point $(x_0, y_0)$ is $y - y_0 = m(x - x_0)$.

Since point B lies on the line that is perpendicular to segment AB at point B, we'll use $(x_0, y_0) = (3, -3)$, which are the coordinates of point B.

Therefore, an equation for the line that is perpendicular to segment AB and passes through point B is

$y - (-3) = \dfrac{9}{4}(x - 3)$, which is $\boxed{y + 3 = \dfrac{9}{4}(x - 3)}$

(b) Find the center and radius of the circle whose equation is $x^2 + y^2 + 8y = 9$.

*Hint:* Complete the square.

**Solution:**

The equation of a circle centered at a point $(h, k)$ with a radius of $r$ is $(x - h)^2 + (y - k)^2 = r^2$.

First, we'll complete the square for $y^2 + 8y$. The coefficient to $y$ is $8$, so we'll divide that number by two, then square the result:

$$\left(\frac{8}{2}\right)^2 = 4^2 = 16$$

So, to complete the square, we add 16 to each side of the original equation, as follows:

$$x^2 + y^2 + 8y = 9$$
$$x^2 + y^2 + 8y + 16 = 9 + 16 = 25$$

Since the only term related to the variable $x$ is $x^2$, which is already a squared term, there is no need to complete the square for the variable $x$.

Therefore, the original equation for the given circle can be expressed as follows, which corresponds to the general form of an equation of a circle:

$$x^2 + (y + 4)^2 = 25$$
$$x^2 + (y - (-4))^2 = 5^2$$

Comparing this result to the general form of an equation of a circle, we see that $h = 0$, $k = -4$, and $r = 5$. Therefore, the center and radius of the given circle are as follows:

Center: $\boxed{(0, -4)}$

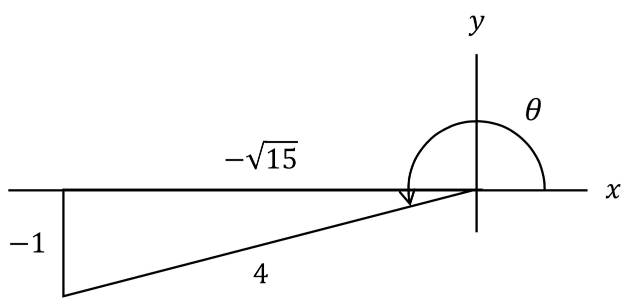
Radius: $\boxed{5}$

3. (20 pts) Parts (a) and (b) are not related to each other.

   (a) If $\csc\theta = -4$ and $\theta$ is on the interval $(\pi, 3\pi/2)$, find the value of $\sec\theta$.

**Solution:**

Since $\theta$ is on the interval $(\pi, 3\pi/2)$, there is a triangle in quadrant III that is associated with angle $\theta$, as shown in the following figure.




Since the triangle is in quadrant III, both the *adjacent* (horizontal) and *opposite* (vertical) components of the triangle are negative.

Since $\csc\theta = -4$, the ratio of the *hypotentuse* component to the *opposite* component is $-4 = 4/(-1)$. Therefore, we can let the *hypotenuse* component equal $4$ and the *opposite* component equal $-1$, as depicted in the preceding figure.

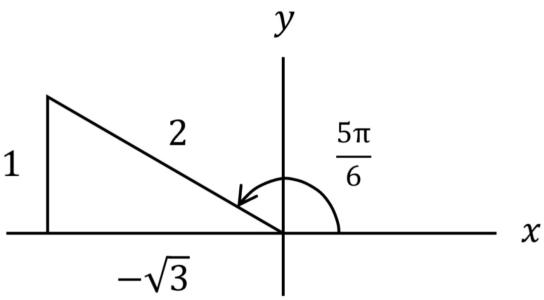
The Pythagorean Theorem indicates that if the length of the hypotenuse of a right triangle is $4$ and the length of one leg of that right triangle is $1$, then the length of the other leg is $\sqrt{4^2 - 1^2} = \sqrt{16-1} = \sqrt{15}$. Since the *adjacent* component of the triangle in the preceding figure is negative, then the value of the *adjacent* component is $-\sqrt{15}$, as shown in the figure.

Therefore, $\sec\theta = \dfrac{\textit{hyponenuse}}{\textit{adjacent}} = \dfrac{4}{-\sqrt{15}} = \boxed{-\dfrac{4}{\sqrt{15}}}$

(b) Evaluate $\tan\left(\frac{5\pi}{6}\right)$

**Solution:**

Since $\frac{\pi}{2} < \frac{5\pi}{6} < \pi$, the angle $\frac{5\pi}{6}$ lies in Quadrant II, as drawn in the following figure.




The reference angle is $\pi - \frac{5\pi}{6} = \frac{\pi}{6}$, which is the $30°$ angle in the special $30° - 60° - 90°$ right triangle. The dimensions of such a triangle are proportional to $1$, $\sqrt{3}$, and $2$, which leads to the set of dimensions displayed in the figure, with the negative value of the *adjacent* component of the triangle and the positive value of the *opposite* component of the triangle accounting for the fact that the triangle is in Quadrant II.

It follows from the figure that $\tan\left(\frac{5\pi}{6}\right) = \frac{\textit{opposite}}{\textit{adjacent}} = \frac{1}{-\sqrt{3}} = \boxed{-1/\sqrt{3}}$

4. (16 pts) Parts (a) and (b) are not related to each other.

(a) Use the trigonometric identity for $\cos(\alpha - \beta)$ and the fact that $15° = 45° - 30°$ to find the exact value of $\cos(15°)$.

**Solution:**

$$\cos(15°) = \cos(45° - 30°) = \cos(45°)\cos(30°) + \sin(45°)\sin(30°)$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \boxed{\frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}}$$

(b) For a circle with a radius of 6 inches, what is the area of a circular sector having a central angle of $50°$? Fully simplify your answer and include the correct unit of measurement.

**Solution:**

Accoring to the formula sheet on the final page of the exam, $A = \frac{1}{2}\theta r^2$, where $\theta$ is expressed in radians.

$$\theta = (50°)\left(\frac{\pi \text{ radians}}{180°}\right) = \frac{5\pi}{18} \text{ radians}$$

So, in the equation $A = \frac{1}{2}\theta r^2$, we have $\theta = 5\pi/18$ radians and $r = 6$ inches. Therefore,

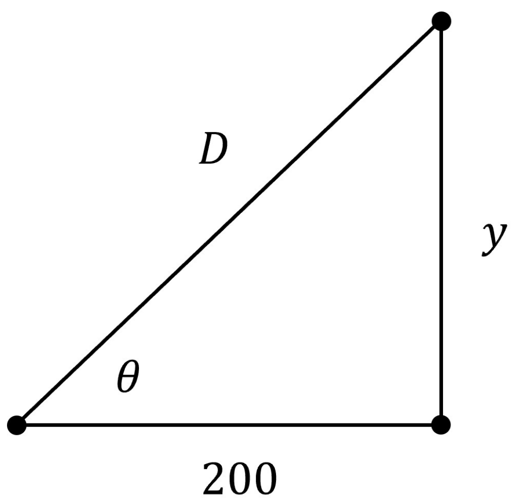
$$A = \frac{1}{2} \cdot \frac{5\pi}{18} \cdot 6^2 = \boxed{5\pi \text{ in}^2}$$

5. (14 pts) Consider a hot-air balloon that is hovering directly above a point A on a flat horizontal field. Suppose that you are lying on another point B on that same field, where the horizontal distance between points A and B is 200 feet. Use the following variable assignments:

- Let $y$ represent the vertical height, in feet, of the balloon above point A.
- Let $D$ represent the straight-line (diagonal) distance, in feet, between point B and the balloon.
- Let $\theta$ represent the angle, in radians, between the field and your line of sight from point B to the balloon.

(a) Draw a right triangle to depict the situation, including correct labels for the angle $\theta$ and the corresponding side lengths $y$, $D$, and 200 of the triangle.

**Solution:**




(b) Find an expression for $y$ in terms of $D$. Your expression should not include the variable $\theta$.

**Solution:**

The Pythagorean Theorem can be applied to the preceding right triangle as follows:

$$200^2 + y^2 = D^2$$

$$y^2 = D^2 - 200^2$$

Therefore, $\boxed{y = \sqrt{D^2 - 200^2} = \sqrt{D^2 - 40000}}$

(In the final step, only the positive root makes physical sense in this problem.)

(c) Find an expression for $y$ in terms of $\theta$. Your expression should include a trigonometric function and it should not include the variable $D$.

**Solution:**

According to the figure above, $\tan\theta = \frac{\textit{opposite}}{\textit{adjacent}} = \frac{y}{200}$.

Therefore, $\boxed{y = 200\tan\theta}$