Remember to write your name! You are **not** allowed to use a calculator, the textbook, your notes, the internet, or your neighbor. To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise. You may quote any relevant theorem from the textbook or from the lectures: Don't re-prove theorems from class unless specifically asked to do so. Everything is real-valued unless specified otherwise. You must do problem #1. Choose two of the three remaining problems. If you submit answers to all problems, problems 1–3 will be graded. All problems are worth 33 points and you get 1 point just for turning in the exam.

Name:

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1. Consider the fixed-point iteration $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \omega \boldsymbol{f}(\boldsymbol{x}_k)$ for solving $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$. Suppose that there is an $\boldsymbol{\alpha}$ such that $\boldsymbol{f}(\boldsymbol{\alpha}) = \boldsymbol{0}$, and that the Jacobian $\mathbf{J}$ of $\boldsymbol{f}$ satisfies

$$\|\mathbf{J}(\boldsymbol{x})\|_2 \leq B \text{ for all } \boldsymbol{x}$$

and

$$\min_{\|\boldsymbol{u}\|_2=1} \boldsymbol{u}^T \left(\mathbf{J} + \mathbf{J}^T\right) \boldsymbol{u} \geq C > 0 \text{ for all } \boldsymbol{x}.$$

(Note that this last condition is that the smallest eigenvalue of the symmetric part of $\mathbf{J}$ is greater than or equal to $(C/2) > 0$.) Prove that if $\omega = C/(2B^2)$ then the iteration is locally convergent in the vicinity of the root.

2. Suppose that $\mathbf{A}$ is strictly diagonally dominant with positive diagonal entries and nonpositive off-diagonal entries. The Jacobi iteration is convergent for this matrix (you do not need to prove this).

   (a) (25 points) Use the Jacobi iteration to express $\mathbf{A}^{-1}$ as a series, i.e. an infinite sum of matrices.

   (b) (8 points) Use (a) to show that the entries of $\mathbf{A}^{-1}$ are non-negative.

3. Let $x_0 < x_1 < \ldots < x_n$. Suppose that p and q are polynomials of degree at most $n-1$ such that

$$p(x_{j-1}) = f(x_{j-1}) \text{ for } j = 1, \ldots, n \text{ and } \quad q(x_j) = f(x_j), \text{ for } j = 1, \ldots, n$$

for some function f whose n^{th} derivative is positive and continuous for $x \in [x_0, x_n]$. Prove that $f(x)$ is between $p(x)$ and $q(x)$ for all $x \in [x_0, x_n]$.

4. Suppose that

$$f(x) = \sum_{m=-M}^{M} c_m e^{\mathrm{i}mx}$$

for some $M > 0$.

(a) Write the formula for the equispaced composite Trapezoid Rule approximation to the integral

$$c_m = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-\mathrm{i}mx} \mathrm{d}x$$

with $N + 1$ points from $x_0 = 0$ to $x_N = 2\pi$.

(b) Suppose that M is even. What is the minimum value of N needed to guarantee that the quadrature from (a) computes the coefficients c_m exactly for

$$-\frac{M}{2} \leq m \leq \frac{M}{2}?$$