Department of Applied Mathematics
MATHEMATICAL STATISTICS PRELIMINARY EXAMINATION
January 2025

Instructions:

- Do **four of the five problems**.
- Place an **X** on the line next to the problem number you are **NOT** submitting for grading.
- Do not write your name anywhere on this exam.
- Write your student number **on each page** submitted for grading.
- Show all relevant work – correct answers without justification will receive no credit!

1. ________ 2. ________ 3. ________ 4. ________ 5. ________ Total ________

Student Number ________________________

Problem 1. (25 points)

Let $Z \sim U(0,1)$ and $X_n = 2^n \mathbb{1}_{[0,1/n)}(Z)$.

(a) Determine, and sketch, the pmf of X_n.

(b) Show $\mathbb{E}(|X_n|^r) \to \infty$ for any $r > 0$.

(c) Show $X_n \xrightarrow{P} 0$.

Problem 2. (25 points)

Let N be the trial number on which we see the first success in a sequence of iid *Bernoulli*(p) trials, $X_1, X_2, \ldots$.

(a) Derive the pmf for N. Note N is called a *Geometric*(p) random variable.

(b) Find the conditional distributions $[N \mid X_1 = 0]$ and $[N \mid X_1 = 1]$.

(c) Use (b) to find $\mathbb{E}N$ and $\text{Var}N$ using the law of iterated expectation.

(d) Let M have the same distribution as N, and be independent of N. Find the pmf of $U = \min(M, N)$.

Problem 3. (25 points)

Consider $X_1, \dots, X_n$ iid from

$$f(x;\theta) = \frac{1}{6\theta^4} x^3 e^{-x/\theta} \mathbb{1}_{(0,\infty)}(x).$$

(a) Can the joint pdf, $f(\mathbf{x};\theta)$, be written in exponential format? Explain why or why not.

(b) Find a complete sufficient statistic $Y = g(X_1, \dots, X_n)$ for θ.

(c) Find a function $\phi(Y)$ that is the UMVUE for θ; is this estimator unique?

(d) Find the variance of your estimator $\phi(Y)$ from part (c).

Problem 4. (25 points)

Suppose we have m different populations, and n samples from each of these populations. In particular, let $Y_{ij} \sim N(\mu_i, \sigma^2)$ for $i = 1, \dots, m$ and $j = 1, \dots, n$ with $\{Y_{ij}\}_{i,j}$ mutually independent.

(a) Assuming $\{\mu_i\}$ are known, find the maximum likelihood estimator (MLE) for σ^2.

(b) Under the setup of part (a), find the Cramér-Rao lower bound on the variance of all unbiased estimators.

(c) Assuming $\{\mu_i\}$ are unknown, find the MLE for σ^2.

(d) Under the setup of part (c), show that, for a fixed n, with $m \to \infty$, the MLE is inconsistent for σ^2 by showing its bias does not vanish.

Problem 5. (25 points)

Let $X_1, \dots, X_n$ be iid $f(x;\theta) = \theta(1-x)^{\theta-1}\mathbb{1}_{(0,1)}(x)$ for $\theta > 0$.

(a) Find the distribution of $Y_i = -\log(1 - X_i)$.

(b) Find the form of the UMP for testing $H_0 : \theta = 1$ versus $H_1 : \theta > 1$, and evaluate the rejection constant for a level α test in terms of a known cdf.

(c) Show that there is no UMP for testing $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$.
