Preliminary Exam
Partial Differential Equations
9AM - 12PM, Thursday, Jan 9, 2025

Student ID (do NOT write your name):

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#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are five problems. **Solve four of the five problems.**
Each problem is worth 25 points.
A sheet of convenient formulae is provided.

1. **Method of Characteristics.**

   (a) (16 points) Solve the Cauchy problem

$$\begin{aligned} t\partial_t u + (x-t)\partial_x u &= u^2, && t>1, \quad x\in\mathbb{R}, \\ u(x,1) &= x, && x\in\mathbb{R}. \end{aligned}$$

You may reference the provided table of ODEs to determine solutions of any that arise.
Solution: We parameterize the initial conditions of $(x(\tau;s), t(\tau;s), u(\tau;s))$ as

$$x(0;s) = s, \quad t(0;s) = 1, \quad u(0;s) = s, \qquad s\in\mathbb{R}.$$

The characteristic equations are then

$$\begin{aligned} \frac{dt}{d\tau} &= t, && t(0;s) = 1, \\ \frac{dx}{d\tau} &= x - t, && x(0;s) = s, \\ \frac{dz}{d\tau} &= z^2, && z(0;s) = s. \end{aligned}$$

Before solving, check for transversality. At $\tau = 0$, we have

$$\begin{vmatrix} \frac{\partial x}{\partial \tau} & \frac{\partial t}{\partial \tau} \\ \frac{\partial x}{\partial s} & \frac{\partial t}{\partial s} \end{vmatrix} = \begin{vmatrix} s-1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0,$$

guaranteeing a solution around the initial data.
Now we solve the characteristic equations. The first equation gives

$$t(\tau, s) = e^\tau.$$

The second equation becomes $dx/d\tau = x - e^\tau$, a linear non-homogeneous equation. The homogeneous solution is $x_h = Ce^\tau$, while a particular solution is $x_p = -\tau e^\tau$. Thus we get $x(\tau,s) = Ce^\tau - \tau e^\tau$. Requiring $x(0,s) = s$, we get

$$x(\tau, s) = se^\tau - \tau e^\tau.$$

Finally, solving for $z(\tau, s)$ we get

$$z(\tau, s) = \frac{s}{1 - s\tau}.$$

Combining the equations for x and t we find that

$$x = st - t\ln(t),$$

so $s = x/t + \ln(t)$. Therefore

$$u(x,t) = z(\tau(x,t), s(x,t)) = z(\tau, s) = \frac{x/t + \ln(t)}{1 - [x/t + \ln(t)]\ln(t)} = \frac{x + t\ln(t)}{t - x\ln(t) - t\ln(t)^2}.$$

(b) (9 points) Determine the region $(x,t) \in \mathcal{R}$ where your solution in part (a) is classical.

Solution: Note first, the solution satisfies the initial condition since $\lim_{t\to 1^+} u(x,t) = x$. It is also valid for $t - x\ln(t) - t\ln(t)^2 > 0$ or $-\infty < x < t \cdot \left[1 - \ln(t)^2\right] / \ln(t)$. The solution is sufficiently differentiable where u_t and u_x can be define where

$$u_t = \frac{t - x + x^2/t + 2x\ln(t) + t\ln(t)^2}{[t - x\ln(t) - t\ln(t)^2]^2}, \qquad u_x = \frac{t}{[t - x\ln(t) - t\ln(t)^2]^2}.$$

The condition on differentiability is thus the same as continuity. Moreover, notice

$$tu_t + (x-t)u_x = \frac{x^2 + 2xt\ln(t) + t^2\ln(t)^2}{[t - x\ln(t) - t\ln(t)^2]^2} = u^2,$$

as expected. Thus, we require $(x,t) \in \mathcal{R} \equiv \{(x,t) \in \mathbb{R}^2 | x < t \cdot \left[1 - \ln(t)^2\right] / \ln(t) \;\&\; t > 1\}$.

ODE	**General Solution**
$ax' + bx = c + dt + et^2, \quad b \neq 0$	$\frac{2a^2e - abd + b^2c}{b^3} + \frac{(bd - 2ae)}{b^2}t + \frac{e}{b}t^2 + Ke^{-\frac{bt}{a}}$
$ax' + bx = e^{ct}, \quad c \neq -b/a$	$\frac{e^{ct}}{ac+b} + Ke^{-\frac{bt}{a}}$
$ax' + bx = e^{ct}, \quad c = -b/a$	$Ke^{-\frac{bt}{a}} + \frac{1}{a}te^{-\frac{bt}{a}}$
$ax' + bx = \cos(ct)$	$\frac{ac\sin(ct) + b\cos(ct)}{a^2c^2 + b^2} + Ke^{-\frac{bt}{a}}$
$ax' + bx = \sin(ct)$	$-\frac{ac\cos(ct) - b\sin(ct)}{a^2c^2 + b^2} + Ke^{-\frac{bt}{a}}$

Table 1: Some first-order linear non-homogeneous ODEs with constant coefficients.

2. **Heat Equation.** Consider the forced heat equation in one dimension,

$$\begin{aligned} \partial_t u &= \partial_{xx} u + f(x,t), & t > 0, \quad x \in \mathbb{R}, \\ u(x,0) &= 0, & x \in \mathbb{R}. \end{aligned}$$

(a) (9 points) Determine $u(x,t)$ in terms of an integral involving $f(x,t)$.

Solution: Using Duhamel's principle, we transform the forced problem into the family of unforced initial value problems

$$\partial_t \tilde{u}(x,t;s) = \partial_{xx} \tilde{u}(x,t;s), \tag{1}$$

$$\tilde{u}(x,s;s) = f(x,s). \tag{2}$$

Each one of these can be solved using the fundamental solution, giving

$$\tilde{u}(x,t;s) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{\frac{-(x-y)^2}{4(t-s)}} f(y,s) dy. \tag{3}$$

Following Duhamel's principle, the solution to the forced problem is obtained by

$$u(x,t) = \int_0^t \tilde{u}(x,t;s) ds = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{\frac{-(x-y)^2}{4(t-s)}} f(y,s) dy ds. \tag{4}$$

(b) (8 points) Calculate the integral for $u(x,t)$ when $f(x,t) = x^2$.

Solution: From (a), we have

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{(x-y)^2}{4(t-s)}} y^2 dy ds.$$

Focusing on the inner integral, we make the change of variables $z = (y-x)/(2\sqrt{t-s})$, so then $y = x + 2\sqrt{t-s}z$ and $dy = 2\sqrt{t-s}dz$ so

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{(x-y)^2}{4(t-s)}} y^2 dy &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-z^2} \left[x^2 + 4\sqrt{t-s} xz + 4(t-s) z^2 \right] dz \\ &= x^2 + 2(t-s) \end{aligned}$$

using the formulas from the sheet. Finally, integrating we obtain

$$u(x,t) = \int_0^t [x^2 + 2(t-s)] ds = tx^2 + t^2.$$

(c) (8 points) Suppose $0 \le |f(x,t)| \le F \in \mathbb{R}^+$ if $x \in [-c,c]$ and $f(x,t) \equiv 0$ if $|x| > c$. Show that

$$|u(x,t)| \le KFct^{1/2},$$

for some constant $K > 0$ and all $t > 0$, $x \in \mathbb{R}$.

Solution: From part (a) and the fact that $f(y,s)$ is zero for $|y| > c$, we have

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{\frac{-(x-y)^2}{4(t-s)}} f(y,s) dy ds = \int_0^t \int_{-c}^{c} \frac{1}{\sqrt{4\pi(t-s)}} e^{\frac{-(x-y)^2}{4(t-s)}} f(y,s) dy ds.$$

Now using the triangle inequality and the fact that $e^{\frac{-(x-y)^2}{4(t-s)}} \le 1$ for $s < t$, we get

$$|u(x,t)| \le \int_0^t \int_{-c}^{c} \frac{1}{\sqrt{4\pi(t-s)}} |f(y,s)| dy ds \le \frac{2cF}{\sqrt{4\pi}} \int_0^t (t-s)^{-1/2} ds = \frac{4}{\sqrt{4\pi}} Fct^{1/2},$$

which is what we wanted to show.

3. **Wave Equation.** Consider the radially symmetric initial value problem in $\mathbb{R}^3$:

$$u_{tt}(\mathbf{x},t) = \Delta u(\mathbf{x},t), \qquad \mathbf{x} \equiv (x_1,x_2,x_3) \in \mathbb{R}^3, \qquad t>0,$$
$$u(\mathbf{x},0) = \phi(|\mathbf{x}|), \quad u_t(\mathbf{x},0) = \psi(|\mathbf{x}|), \quad \mathbf{x} \in \mathbb{R}^3, \quad |\mathbf{x}| = \sqrt{x_1^2+x_2^2+x_3^2}.$$

(a) (12 points) Determine the solution $u(\mathbf{x},t) = v(r,t)/r$ by exploiting radial symmetry ($r \equiv |\mathbf{x}| \geq 0$).
Solution: We assume $u(\mathbf{x},t) = v(r,t)/r$ where $r = |\mathbf{x}| \geq 0$ and $\lim_{r\to 0^+} v = 0$ as well as the formula for the Laplacian in radial coordinates in $\mathbb{R}^3$,

$$\frac{1}{r}v_{tt} = \frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\left(\frac{v}{r}\right)\right] = \frac{1}{r^2}\frac{d}{dr}\left[rv_r - v\right] = \frac{1}{r}v_{rr} \quad \Rightarrow \quad v_{tt} = v_{rr}.$$

The problem then becomes an initial boundary value problem on $r>0$ and $t>0$,

$$\begin{aligned} &v_{tt}(r,t) = v_{rr}(r,t), \qquad r>0,\ t>0,\\ &v(r,0) = r\phi(r), \quad v_t(r,0) = r\psi(r), \quad r>0,\\ &v(0,t) = 0, \qquad t>0, \end{aligned}$$

which can be solved using odd reflection of d'Alembert's solution. That is, for $r>t$, we have

$$v(r,t) = \frac{1}{2}\left[(r+t)\phi(r+t) + (r-t)\phi(r-t)\right] + \frac{1}{2}\int_{r-t}^{r+t} s\psi(s)ds,$$

whereas for $0<r<t$, we have

$$v(r,t) = \frac{1}{2}\left[(r+t)\phi(r+t) - (t-r)\phi(t-r)\right] + \frac{1}{2}\int_{t-r}^{r+t} s\psi(s)ds,$$

which we can rewrite using $|\mathbf{x}| = r$ and convert to $u = v/r$, so

$$u(\mathbf{x},t) = \frac{1}{2|\mathbf{x}|}\cdot\begin{cases} (|\mathbf{x}|+t)\phi(|\mathbf{x}|+t) + (|\mathbf{x}|-t)\phi(|\mathbf{x}|-t) + \int_{|\mathbf{x}|-t}^{|\mathbf{x}|+t} s\psi(s)ds, & |\mathbf{x}|>t,\\ (|\mathbf{x}|+t)\phi(|\mathbf{x}|+t) - (t-|\mathbf{x}|)\phi(t-|\mathbf{x}|) + \int_{t-|\mathbf{x}|}^{|\mathbf{x}|+t} s\psi(s)ds, & 0<|\mathbf{x}|<t. \end{cases}$$

(b) (8 points) Assume $\phi>0$ only for $|\mathbf{x}| \in (1,2)$ and $\psi \equiv 0$. Determine the support of u at $t=2$.
Solution: Examining the solution in (a): for $|\mathbf{x}|>2$, we have support where $|\mathbf{x}| \in (3,4)$ and for $|\mathbf{x}| \in (0,2)$, we have support where $|\mathbf{x}| \in (0,1)$, so $\text{supp}[u] = \{|\mathbf{x}| \in (0,1)\cup(3,4)\}$.

(c) (5 points) Find the limit $\lim_{|\mathbf{x}|\to 0^+} u(\mathbf{x},t) = u(\mathbf{0},t)$ and a condition so $u(\mathbf{0},t) \equiv 0$ for all $t \geq 0$.
Solution: Define $r = |\mathbf{x}|$ as before, and require $u(\mathbf{0},t) \equiv 0$ in the limit:

$$\begin{aligned}\lim_{r\to 0^+} u(\mathbf{x},t) &= \lim_{r\to 0^+} \frac{(r+t)\phi(r+t) + (r-t)\phi(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{r+t} s\psi(s)ds\\ &= \phi(t) + t\phi'(t) + t\psi(t) \equiv 0 \qquad \Rightarrow \qquad \phi'(t) = -\psi(t).\end{aligned}$$

4. **Laplace's equation.** Suppose $u \in C^2(\Omega)$ is harmonic ($\Delta u(\mathbf{x}) = 0$) on $\mathbf{x} \in \Omega \subset \mathbb{R}^n$ bounded.

(a) (10 points) Prove u satisfies the mean value property for any ball $B(\mathbf{x}, r) \subset \Omega$:

$$u(\mathbf{x}) = \fint_{\partial B(\mathbf{x},r)} u(\mathbf{y}) dS_{\mathbf{y}} = \fint_{B(\mathbf{x},r)} u(\mathbf{y}) d\mathbf{y}.$$

Solution: Since u is harmonic, then for any $B(\mathbf{x}, r) \subset \Omega$,

$$0 = \int_{B(\mathbf{x},r)} \Delta u(\mathbf{y}) d\mathbf{y} = \int_{\partial B(\mathbf{x},r)} \frac{\partial u}{\partial n} dS_{\mathbf{y}} = \int_{\partial B(\mathbf{x},r)} \nabla u(\mathbf{y}) \cdot \frac{\mathbf{y}-\mathbf{x}}{r} dS_{\mathbf{y}}. \tag{5}$$

Now define $\phi(r) = \fint_{\partial B(\mathbf{x},r)} u(\mathbf{y}) dS_{\mathbf{y}}$, and let $\mathbf{y} = \mathbf{x} + r\mathbf{z}$ with $\mathbf{z} \in B(\mathbf{0}, 1)$ and calculate

$$\phi'(r) = \frac{d}{dr} \fint_{\partial B(\mathbf{0},1)} u(\mathbf{x} + r\mathbf{z}) dS_{\mathbf{z}} = \fint_{\partial B(\mathbf{0},1)} \nabla u(\mathbf{x} + r\mathbf{z}) \cdot \mathbf{z} dS_{\mathbf{z}} = \fint_{\partial B(\mathbf{x},r)} \nabla u(\mathbf{y}) \cdot \frac{\mathbf{y}-\mathbf{x}}{r} dS_{\mathbf{y}} = 0$$

by Eq. (5). Thus, $\phi(r)$ is constant, so $\phi(r) = \lim_{s\to 0^+} \phi(s) = u(\mathbf{x})$. Furthermore,

$$\fint_{B(\mathbf{x},r)} u(\mathbf{y}) d\mathbf{y} = \frac{1}{|B(\mathbf{x}, r)|} \int_0^r \int_{\partial B(\mathbf{x},\rho)} u(\mathbf{y}) dS_{\mathbf{y}} d\rho = u(\mathbf{x}).$$

(b) (8 points) Consider the boundary value problem (BVP) on the unit ball:

$$\begin{aligned} \Delta u(\mathbf{x}) &= 0, & \mathbf{x} &\in B(\mathbf{0}, 1) \subset \mathbb{R}^2, \\ u(\mathbf{x}) &= g(\mathbf{x}), & \mathbf{x} &\in \partial B(\mathbf{0}, 1). \end{aligned}$$

Write the BVP for the Green's function $G(\mathbf{x}, \mathbf{y})$ and use $\Phi(|\mathbf{x} - \mathbf{y}|)$ the associated fundamental solution to construct the Green's function $G(\mathbf{x}, \mathbf{y})$, checking all necessary conditions.

Solution: The BVP for $G(\mathbf{x}, \mathbf{y})$ has form

$$\begin{aligned} -\Delta G(\mathbf{x}, \mathbf{y}) &= \delta(\mathbf{x} - \mathbf{y}), & \mathbf{x}, \mathbf{y} &\in B(\mathbf{0}, 1), \\ G(\mathbf{x}, \mathbf{y}) &= 0, & \mathbf{x} &\in \partial B(\mathbf{0}, 1), \quad \mathbf{y} \in B(\mathbf{0}, 1). \end{aligned}$$

We take $G(\mathbf{x}, \mathbf{y}) = \Phi(|\mathbf{x} - \mathbf{y}|) - \Phi(|\mathbf{x}| \cdot |\hat{\mathbf{x}} - \mathbf{y}|)$ where $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|^2$, so clearly $\hat{\mathbf{x}} \notin B(\mathbf{0}, 1)$ if $\mathbf{x} \in B(\mathbf{0}, 1)$ and if $|\mathbf{x}| = 1$ then $|\mathbf{x}| \cdot |\hat{\mathbf{x}} - \mathbf{y}| = |\mathbf{x} - \mathbf{y}|$, so $G(\mathbf{x}, \mathbf{y}) \equiv 0$.

(c) (7 points) Prove that the solution to the BVP in (b) has the form:

$$u(\mathbf{x}) = -\int_{\partial B(\mathbf{0},1)} \frac{\partial G}{\partial n_{\mathbf{y}}} g(\mathbf{y}) dS_{\mathbf{y}}.$$

Solution: Note by the Divergence theorem, we can write

$$\int_{B(\mathbf{0},1)} \left[\Delta u(\mathbf{y}) G(\mathbf{x}, \mathbf{y}) - \Delta_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) u(\mathbf{y})\right] d\mathbf{y} = \int_{\partial B(\mathbf{0},1)} \left[\frac{\partial u}{\partial n_{\mathbf{y}}}(\mathbf{y}) G(\mathbf{x}, \mathbf{y}) - \frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) u(\mathbf{y})\right] dS_{\mathbf{y}},$$

and since $\Delta u = 0$ and $-\Delta_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$ and boundary conditions, we have

$$\int_{B(\mathbf{0},1)} \delta(\mathbf{x} - \mathbf{y}) u(\mathbf{y}) d\mathbf{y} = u(\mathbf{x}) = \int_{\partial B(\mathbf{0},1)} \left[\frac{\partial u}{\partial n_{\mathbf{y}}}(\mathbf{y}) \cdot 0 - \frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) g(\mathbf{y})\right] dS_{\mathbf{y}} = -\fint_{\partial B(\mathbf{0},1)} \frac{\partial G}{\partial n_{\mathbf{y}}} g(\mathbf{y}) dS_{\mathbf{y}}.$$

5. **Separation of Variables.** Consider the initial boundary value problem (IBVP) on the unit interval:

$$\begin{aligned} u_t(x,t) + u(x,t) &= u_{xx}(x,t) + 1, && x \in (0,1), \quad t > 0, \\ u(0,t) &= 0, \qquad u_x(1,t) = e^{-1}, && t > 0, \\ u(x,0) &= f(x), && x \in (0,1). \end{aligned}$$

(a) (5 points) Find the steady-state solution $\bar{u}(x) = \lim_{t\to\infty} u(x,t)$.

Solution: The steady-state ODE $-\bar{u}'' + \bar{u} = 1$ has general solution $\bar{u} = 1 + Ae^{x} + Be^{-x}$, so with boundary conditions $\bar{u}(0) = 0$ and $\bar{u}'(1) = e^{-1}$, we find $\bar{u}(x) = 1 - e^{-x}$.

(b) (10 points) Formulate the IBVP for $v(x,t) = u(x,t) - \bar{u}(x)$. Solve for $v(x,t)$ using separation of variables. Then formulate $u(x,t) = v(x,t) + \bar{u}(x)$.

Solution:

$$\begin{aligned} v_t(x,t) + v(x,t) &= v_{xx}(x,t), && x \in (0,1), \quad t > 0, \\ v(0,t) &= 0, \qquad v_x(1,t) = 0, && t > 0, \\ v(x,0) &= f(x) + e^{-x} - 1, && x \in (0,1). \end{aligned}$$

Separable solutions then satisfy

$$X(x)T'(t) + X(x)T(t) = X''(x)T(t) \quad \Rightarrow \quad \frac{T'(t) + T(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

so $T'(t) = -(\lambda + 1)T(t)$ and $X''(x) + \lambda X(x) = 0$ with $X(0) = X'(1) = 0$.

Note: The factor of one could also end up in the X BVP but it is messier.

First solve the X eigenvalue problem. If $\lambda \leq 0$, BCs always force $X \equiv 0$.
If $\lambda_n = \mu_n^2 > 0$, $X_n(x) = a_n \cos(\mu_n x) + b_n \sin(\mu_n x)$ and $X_n(0) = 0$ forces $a_n = 0$, while $X_n'(1) = b_n \mu_n \cos(\mu_n) = 0$ or $\mu_n = \frac{\pi}{2} + n\pi$ for $n = 0, 1, 2, \ldots$, so $\lambda_n = \left[\frac{\pi}{2} + n\pi\right]^2$. Note $n = -1, -2, -3, \ldots$ are redundant as they would merely change the sign of the sin.
We thus have $T_n(t) = e^{-(\lambda_n + 1)t}$, and we can write a general solution to the original IBVP as

$$u(x,t) = 1 - e^{-x} + \sum_{n=0}^{\infty} b_n e^{-(\lambda_n+1)t} \sin(\mu_n x), \qquad b_n = 2\int_0^1 \sin(\mu_n x)(f(x) + e^{-x} - 1)\,dx, \quad n = 0, 1, 2, \ldots$$

(c) (10 points) Use an energy method to show that any classical solution u to the IBVP is unique.

Solution: Consider the energy $E[u](t) = \frac{1}{2}\int_0^1 u(x,t)^2 dx$, so then if both u and w solve the IBVP, then $z(x,t) = u(x,t) - w(x,t)$ satisfies the homogenized IBVP from (b) with $z(x,0) = 0$ for $x \in (0,1)$. We then compute

$$\begin{aligned} \frac{d}{dt}E[z](t) &= \int_0^1 z_t(x,t)z(x,t)\,dx = \int_0^1 z_{xx}(x,t)z(x,t)\,dx - \int_0^1 z(x,t)^2 dx \\ &= \left[z_x(x,t)z(x,t)\right]_{x=0}^{x=1} - \int_0^1 z_x(x,t)^2 + z(x,t)^2 dx = -\int_0^1 z_x(x,t)^2 + z(x,t)^2 dx. \end{aligned}$$

Boundary terms vanish due to the IBVP, so $\frac{dE}{dt}[z] \leq 0$. Thus, $E[z](t) \equiv 0$ for $t > 0$ since $E[z] \geq 0$ and $E[z](0) \equiv 0$ due to the IBVP. Then by continuity we know $z \equiv 0$ so $u(x,t) \equiv w(x,t)$ for all $x \in (0,1)$ and $t > 0$.