Preliminary Exam
Partial Differential Equations
9AM - 12PM, Thursday, Jan 9, 2025

Student ID (do NOT write your name):

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#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are five problems. **Solve four of the five problems.**
Each problem is worth 25 points.
A sheet of convenient formulae is provided.

1. **Method of Characteristics.**

   (a) (16 points) Solve the Cauchy problem

$$\begin{aligned} t\partial_t u + (x-t)\partial_x u &= u^2, && t > 1, \quad x \in \mathbb{R}, \\ u(x,1) &= x, && x \in \mathbb{R}. \end{aligned}$$

   You may reference the provided table of ODEs to determine solutions of any that arise.

   (b) (9 points) Determine the region $(x,t) \in \mathcal{R}$ where your solution in part (a) is classical.

ODE	**General Solution**
$ax' + bx = c + dt + et^2, \quad b \neq 0$	$\frac{2a^2e-abd+b^2c}{b^3} + \frac{(bd-2ae)}{b^2}t + \frac{e}{b}t^2 + Ke^{-\frac{bt}{a}}$
$ax' + bx = e^{ct}, \quad c \neq -b/a$	$\frac{e^{ct}}{ac+b} + Ke^{-\frac{bt}{a}}$
$ax' + bx = e^{ct}, \quad c = -b/a$	$Ke^{-\frac{bt}{a}} + \frac{1}{a}te^{-\frac{bt}{a}}$
$ax' + bx = \cos(ct)$	$\frac{ac\sin(ct)+b\cos(ct)}{a^2c^2+b^2} + Ke^{-\frac{bt}{a}}$
$ax' + bx = \sin(ct)$	$-\frac{ac\cos(ct)-b\sin(ct)}{a^2c^2+b^2} + Ke^{-\frac{bt}{a}}$

Table 1: Some first-order linear non-homogeneous ODEs with constant coefficients.

2. **Heat Equation.** Consider the forced heat equation in one dimension,

$$\begin{aligned} \partial_t u &= \partial_{xx} u + f(x,t), \qquad && t > 0, \quad x \in \mathbb{R}, \\ u(x,0) &= 0, && x \in \mathbb{R}. \end{aligned}$$

(a) (9 points) Determine $u(x,t)$ in terms of an integral involving $f(x,t)$.

(b) (8 points) Calculate the integral for $u(x,t)$ when $f(x,t) = x^2$.

(c) (8 points) Suppose $0 \le |f(x,t)| \le F \in \mathbb{R}^+$ if $x \in [-c, c]$ and $f(x,t) \equiv 0$ if $|x| > c$. Show that

$$|u(x,t)| \le KFct^{1/2},$$

for some constant $K > 0$ and all $t > 0$, $x \in \mathbb{R}$.

3. **Wave Equation.** Consider the radially symmetric initial value problem in $\mathbb{R}^3$:

$$\begin{aligned} &u_{tt}(\mathbf{x},t) = \Delta u(\mathbf{x},t), && \mathbf{x} \equiv (x_1, x_2, x_3) \in \mathbb{R}^3, \qquad t > 0, \\ &u(\mathbf{x},0) = \phi(|\mathbf{x}|), \quad u_t(\mathbf{x},0) = \psi(|\mathbf{x}|), && \mathbf{x} \in \mathbb{R}^3, \quad |\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}. \end{aligned}$$

(a) (12 points) Determine the solution $u(\mathbf{x},t) = v(r,t)/r$ by exploiting radial symmetry ($r \equiv |\mathbf{x}| \ge 0$).

(b) (8 points) Assume $\phi > 0$ only for $|\mathbf{x}| \in (1,2)$ and $\psi \equiv 0$. Determine the support of u at $t = 2$.

(c) (5 points) Find the limit $\lim_{|\mathbf{x}| \to 0^+} u(\mathbf{x},t) = u(\mathbf{0},t)$ and a condition so $u(\mathbf{0},t) \equiv 0$ for all $t \ge 0$.

4. **Laplace's equation.** Suppose $u \in C^2(\Omega)$ is harmonic ($\Delta u(\mathbf{x}) = 0$) on $\mathbf{x} \in \Omega \subset \mathbb{R}^n$ bounded.

   (a) (10 points) Prove u satisfies the mean value property for any ball $B(\mathbf{x}, r) \subset \Omega$:

   $$u(\mathbf{x}) = \fint_{\partial B(\mathbf{x},r)} u(\mathbf{y}) dS_{\mathbf{y}} = \fint_{B(\mathbf{x},r)} u(\mathbf{y}) d\mathbf{y}.$$

   (b) (8 points) Consider the boundary value problem (BVP) on the unit ball:

   $$\begin{aligned} \Delta u(\mathbf{x}) &= 0, && \mathbf{x} \in B(\mathbf{0}, 1) \subset \mathbb{R}^2, \\ u(\mathbf{x}) &= g(\mathbf{x}), && \mathbf{x} \in \partial B(\mathbf{0}, 1). \end{aligned}$$

   Write the BVP for the Green's function $G(\mathbf{x}, \mathbf{y})$ and use $\Phi(|\mathbf{x} - \mathbf{y}|)$ the associated fundamental solution to construct the Green's function $G(\mathbf{x}, \mathbf{y})$, checking all necessary conditions.

   (c) (7 points) Prove that the solution to the BVP in (b) has the form:

   $$u(\mathbf{x}) = -\int_{\partial B(\mathbf{0},1)} \frac{\partial G}{\partial n_{\mathbf{y}}} g(\mathbf{y}) dS_{\mathbf{y}}.$$

5. **Separation of Variables.** Consider the initial boundary value problem (IBVP) on the unit interval:

   $$\begin{aligned} u_t(x,t) + u(x,t) &= u_{xx}(x,t) + 1, && x \in (0,1), \quad t > 0, \\ u(0,t) = 0, \qquad & u_x(1,t) = e^{-1}, && t > 0, \\ u(x,0) &= f(x), && x \in (0,1). \end{aligned}$$

   (a) (5 points) Find the steady-state solution $\bar{u}(x) = \lim_{t\to\infty} u(x,t)$.

   (b) (10 points) Formulate the IBVP for $v(x,t) = u(x,t) - \bar{u}(x)$. Solve for $v(x,t)$ using separation of variables. Then formulate $u(x,t) = v(x,t) + \bar{u}(x)$.

   (c) (10 points) Use an energy method to show that any classical solution u to the IBVP is unique.