Write your name below. This exam is worth 100 points. You must show all your work to receive credit on each problem and must fully simplify your answers unless otherwise instructed. You are allowed to use one page of notes. You are not allowed to use a calculator or any computational software.

Name:

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1. (18 points) The following questions are unrelated.

   (a) (9 points) Set up but do not evaluate an integral to find the volume of the solid below the function $f(x, y) = xy^2$ and above the triangle in the xy-plane with vertices $(1, 0)$, $(2, 2)$, and $(3, 0)$

   Solution:

$$V = \int_0^2 \int_{1+y/2}^{3-y/2} \int_0^{xy^2} dzdxdy$$

or

$$V = \int_1^2 \int_0^{2x-2} \int_0^{xy^2} dzdydx + \int_2^3 \int_0^{6-2x} \int_0^{xy^2} dzdydx$$

   (b) (9 points) Let D be the region in Q1 of the xy-plane bounded by the lines $y = \frac{1}{\sqrt{3}}x$ and $y = \sqrt{3}x$ and the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$. Convert the following integral into polar coordinates and evaluate:

$$\int\int_D \frac{1}{(x^2 + y^2)} dA$$

   Solution:

$$\begin{aligned}
\int\int_D \frac{1}{(x^2 + y^2)} dA &= \int_{\pi/6}^{\pi/3} \int_1^{\sqrt{2}} \frac{1}{r^2} r dr d\theta \\
&= \left(\int_{\pi/6}^{\pi/3} d\theta \right) \left(\int_1^{\sqrt{2}} \frac{1}{r} dr \right) \\
&= \left(\frac{\pi}{6} \right) \left(\ln(\sqrt{2}) \right) \\
&= \frac{\pi}{12} \ln(2)
\end{aligned}$$

2. (25 points) Consider the volume defined by the integral $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{4-\sqrt{x^2+y^2}} dzdydx.$

   (a) (10 points) Set up an integral to calculate the volume in cylindrical coordinates.

   Solution:

$$V = \int_0^{2\pi}\int_0^2\int_r^{4-r} rdzdrd\theta$$

   (b) (10 points) Set up an integral to calculate the volume in spherical coordinates.

   Solution:

$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{4/(\cos\phi+\sin\phi)} \rho^2\sin\phi d\rho d\phi d\theta$$

   (c) (5 points) Find the volume by evaluating one of these integrals.

   Solution:

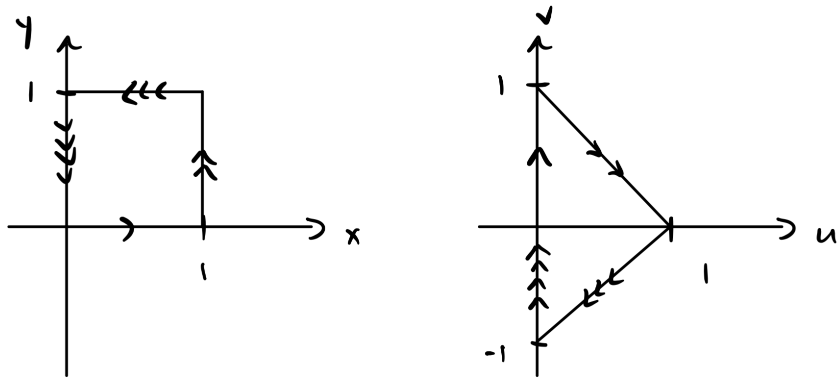
$$\begin{aligned}\int_0^{2\pi}\int_0^2\int_r^{4-r} rdzdrd\theta &= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^2\int_r^{4-r} rdzdr\right)\\ &= 2\pi\int_0^2 4r-2r^2dr\\ &= 2\pi\left(2r^2-2\frac{r^3}{3}\right)\Big|_0^2\\ &= \frac{16\pi}{3}\end{aligned}$$

3. (18 points) Let the function $f(x,y) = (x+y)e^{x^2y-xy^2}$, the region $D = \{(x,y)|0 \leq x \leq 1, 0 \leq y \leq 1\}$, and T be a coordinate transformation given by

$$u(x,y) = xy$$
$$v(x,y) = x - y$$

(a) (5 points) Sketch the image of D in the uv-plane, labeling all relevant points.

Solution:



(b) (5 points) Solve for x and y in terms of u and v.

Solution: We solve for x in the second equation and substitute into the first, which we can then solve for y:

$$\begin{aligned} x &= v + y \\ u &= (v+y)y \\ 0 &= y^2 + vy - u \\ y &= \frac{1}{2}\left(-v \pm \sqrt{v^2+4u}\right) \end{aligned}$$

Since $y \geq 0$, we take the plus sign:

$$y = \frac{1}{2}\left(-v + \sqrt{v^2+4u}\right)$$

Since $x = v + y$ we then have

$$x = \frac{1}{2}\left(v + \sqrt{v^2+4u}\right)$$

(c) (3 points) Find the Jacobian for the transformation.

Solution:

$$\frac{\partial x}{\partial u} = \frac{2}{\sqrt{v^2+4u}}$$
$$\frac{\partial y}{\partial u} = \frac{2}{\sqrt{v^2+4u}}$$
$$\frac{\partial x}{\partial v} = \frac{1}{2}\left(1+\frac{v}{\sqrt{v^2+4u}}\right)$$
$$\frac{\partial y}{\partial v} = \frac{1}{2}\left(-1+\frac{v}{\sqrt{v^2+4u}}\right)$$

$$\begin{aligned} J &= \left|\frac{2}{\sqrt{v^2+4u}}\frac{1}{2}\left(-1+\frac{v}{\sqrt{v^2+4u}}\right) - \frac{2}{\sqrt{v^2+4u}}\frac{1}{2}\left(1+\frac{v}{\sqrt{v^2+4u}}\right)\right| \\ &= \frac{2}{\sqrt{v^2+4u}} \end{aligned}$$

(d) (5 points) Set up the integral of f over the image of D in the uv-plane. Do not evaluate the integral, but fully simplify the integrand.

Solution:

$$\begin{aligned} \int_0^1\int_0^1 (x+y)e^{x^2y-xy^2}\,dxdy &= \int_0^1\int_{u-1}^{1-u}\left(\frac{1}{2}\left(v+\sqrt{v^2+4u}\right)+\frac{1}{2}\left(-v+\sqrt{v^2+4u}\right)\right)e^{uv}\frac{2}{\sqrt{v^2+4u}}dvdu \\ &= \int_0^1\int_{u-1}^{1-u} 2e^{uv}\,dvdu \end{aligned}$$

4. (20 points) In quantum physics, the exact location of a particle becomes impossible to predict, and instead we have a probability density of finding a particle at location (x, y). Suppose this is given by

$$p(x,y) = A\sin^2(2\pi x)\sin(\pi y) \qquad 0 \le x \le 1\ ,\ 0 \le y \le 1$$

where A is a constant.

(a) (5 points) What must the value of A be?

Solution: We know that the integral over the domain must equal 1:

$$\begin{aligned}
1 &= \int_0^1 \int_0^1 A\sin^2(2\pi x)\sin(\pi y)dxdy \\
1 &= A\left(\int_0^1 \sin^2(2\pi x)dx\right)\left(\int_0^1 \sin(\pi y)dy\right) \\
1 &= A\left(\int_0^1 \frac{1-\cos(4\pi x)}{2}dx\right)\left(\frac{-\cos(\pi y)}{\pi}\Big|_0^1\right) \\
1 &= A\left(\frac{1}{2}\right)\left(\frac{2}{\pi}\right) \\
A &= \pi
\end{aligned}$$

(b) (5 points) What is the probability of finding the particle in the region defined by $\frac{1}{2} \le x \le \frac{3}{4}$ and $\frac{1}{2} \le y \le 1$?

Solution:

$$\begin{aligned}
P &= \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^{\frac{3}{4}} \pi\sin^2(2\pi x)\sin(\pi y)dxdy \\
P &= \pi\left(\int_{\frac{1}{2}}^1 \sin(\pi y)dy\right)\left(\int_{\frac{1}{2}}^{\frac{3}{4}} \frac{1-\cos(4\pi x)}{2}dx\right) \\
P &= \pi\left(\frac{-\cos(\pi y)}{\pi}\right)\Big|_{\frac{1}{2}}^1 \left(\frac{1-\cos(4\pi x)}{2}\right)\Big|_{\frac{1}{2}}^{\frac{3}{4}}
\end{aligned}$$

(c) (10 points) Evaluate an integral to find the expected value of y.

5. (10 points) Prove that $||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = 2\left(||\mathbf{u}||^2 + ||\mathbf{v}||^2\right)$ using the laws of dot products.

Solution:

$$\begin{aligned} ||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\cdot \\ &= \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{u} - 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} \\ &= 2\mathbf{u}\cdot\mathbf{u} + 2\mathbf{v}\cdot\mathbf{v} \\ &= 2\left(||\mathbf{u}||^2 + ||\mathbf{v}||^2\right) \end{aligned}$$

6. (9 points, 3 each) Given the matrices

$$A = \begin{pmatrix} -1 & 0 & 4 \\ -2 & -2 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 4 \\ -2 & 1 \\ 0 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$$

compute the following or state why the expression does not exist.

(a) $A + B^T$

Solution:

$$A + B^T = \begin{pmatrix} -1 & 0 & 4 \\ -2 & -2 & 3 \end{pmatrix} + \begin{pmatrix} -1 & -2 & 0 \\ 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} -2 & -2 & 4 \\ 2 & -1 & 5 \end{pmatrix}$$

(b) $AB - C$

Solution:

$$\begin{aligned} AB - C &= \begin{pmatrix} -1 & 0 & 4 \\ -2 & -2 & 3 \end{pmatrix} \begin{pmatrix} -1 & 4 \\ -2 & 1 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 4 \\ 6 & -4 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2 \\ 3 & -9 \end{pmatrix} \end{aligned}$$

(c) BAC

Solution: Since B is 3×2 and A is 2×3 their product BA is 3×3. Since C is 2×2, the product $(BA)C$ does not exist.