APPM 3570/STAT 3100 — Exam 2 — Fall 2024

On the front of your bluebook, write (1) **your name**, (2) **Exam 2**, (3) **APPM 3570/STAT 3100**. Correct answers with no supporting work may receive little or no credit. Books, notes and electronic devices of any kind are not allowed. Your exam should be uploaded to Gradescope in a PDF format (Recommended: **Genius Scan**, **Scannable** or **CamScanner** for iOS/Android). **Show all work, justify your answers.** **Do all problems.** Students are required to re-write the **honor code statement** in the box below on the **first page** of their exam submission and **sign and date it**:

On my honor, as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work. Signature:____________________________ Date:__________

1. [Exam02] (40pts) Answer all the problems below. Justify your answers.

   (a) (10pts) An average of 3 car accidents occur per week on a stretch of highway. Find the probability that there will be 4 car accidents this week given that at least 1 accident has already occurred. Simplify your answer, *no sigma notation* in your final answer.

   (b) (10pts) The *Rockwell hardness* of a metal is determined by impressing a hardened point into the surface of the metal and then measuring the depth of penetration of the point and is measured on a continuous scale. Suppose the Rockwell hardness of a particular alloy is Normally distributed with mean 70 and standard deviation 3. What is the probability that a randomly chosen specimen has its hardness between 67 and 76? (Give your answer in terms of Φ, the cumulative distribution function of the Standard Normal rv.)

   (c) (10pts) Suppose $X \sim \text{Uniform}(0,1)$ with pdf $f_X(x) = \begin{cases} 1, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$ For some fixed $\mu > 0$, define the random variable $W = -\frac{1}{\mu}\ln(1-X)$. Find the *probability density function* of W (define it for all $\mathbb{R}$).

   (d) (10pts) Suppose we roll a pair of fair dice, X is the number of dice that come up 1, and Y is the number of dice that come up 4, 5 or 6. Find the *joint probability mass function* of (X, Y). Define the joint pmf for all values of $\mathbb{R}^2$.

Solution: (a)(10pts) Let X be the number of accidents that occur then $X \sim \text{Poisson}(\lambda = 3)$, now we wish to find $\text{P}(X = 4 | X \geq 1)$, now

$$\text{P}(X = 4|X \geq 1) = \frac{\text{P}(X = 4, X \geq 1)}{\text{P}(X \geq 1)} = \frac{\text{P}(X = 4)}{\text{P}(X \geq 1)} = \frac{\text{P}(X = 4)}{1 - \text{P}(X < 1)} = \frac{\text{P}(X = 4)}{1 - \text{P}(X = 0)}$$

now note $p(k) = e^{-\lambda}\dfrac{\lambda^k}{k!}$, so

$$\text{P}(X = 4|X \geq 1) = \frac{\text{P}(X = 4)}{1 - \text{P}(X = 0)} = \frac{e^{-3}3^4/4!}{1 - e^{-3}} = \frac{e^{-3}3^4}{4!(1 - e^{-3})} = \frac{3^4}{4!(e^3 - 1)}$$

(b)(10pts) Let X be the Rockwell hardness of the particular alloy then $X \sim \text{Normal}(\mu = 70, \sigma^2 = 3^2)$ so

$$P(67 < X < 76) = P\Big(\frac{67 - 70}{3} < \frac{X - \mu}{\sigma} < \frac{76 - 70}{3}\Big) = P(-1 < Z < 2) = \Phi(2) - \Phi(-1).$$

(c)(10pts) If $X \sim \text{Uniform}(0,1)$ then $1 - X \in (0,1)$ so, if $W = -\frac{1}{\mu}\ln(1-X)$, then $W > 0$ thus, for $a \leq 0$, we have $P(W \leq a) = 0$. Now if $a > 0$, then

$$\begin{aligned} F_W(a) = P(W \leq a) = P\Big(-\frac{1}{\mu}\ln(1-X) \leq a\Big) &= P\big(\ln(1-X) \geq -\mu a\big) \\ &= P\big(1 - X \geq e^{-\mu a}\big) \\ &= P\big(X \leq 1 - e^{-\mu a}\big) = \int_0^{1-e^{-\mu a}} 1\, dx = 1 - e^{-\mu a} \end{aligned}$$

so, the cdf of W is

$$F_W(a) = \begin{cases} 1 - e^{-\mu a}, & \text{if } a > 0, \\ 0, & \text{if } a \leq 0, \end{cases} \Rightarrow f_W(a) = \frac{d}{da} F_W(a) = \begin{cases} \mu e^{-\mu a}, & \text{if } a > 0, \\ 0, & \text{if } a \leq 0, \end{cases} \Rightarrow W \sim \text{Exponential}(\mu).$$

(Note, we used the fact that $0 < a < \infty \Rightarrow 0 > -\mu a > -\infty \Rightarrow 1 > e^{-\mu a} > 0 \Rightarrow 0 < 1 - e^{-\mu a} < 1$.)

(d)(10pts) Using the table of dice rolls:

Dice 1 \ Dice 2	1	2	3	4	5	6
1	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$
2	$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$
3	$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$	$(3,5)$	$(3,6)$
4	$(4,1)$	$(4,2)$	$(4,3)$	$(4,4)$	$(4,5)$	$(4,6)$
5	$(5,1)$	$(5,2)$	$(5,3)$	$(5,4)$	$(5,5)$	$(5,6)$
6	$(6,1)$	$(6,2)$	$(6,3)$	$(6,4)$	$(6,5)$	$(6,6)$

we see that $X, Y \in \{0, 1, 2\}$ according to the following table:

(X, Y) Values

Dice 1 \ Dice 2	1	2	3	4	5	6
1	$(X=2, Y=0)$	$(X=1, Y=0)$	$(X=1, Y=0)$	$(X=1, Y=1)$	$(X=1, Y=1)$	$(X=1, Y=1)$
2	$(X=1, Y=0)$	$(X=0, Y=0)$	$(X=0, Y=0)$	$(X=0, Y=1)$	$(X=0, Y=1)$	$(X=0, Y=1)$
3	$(X=1, Y=0)$	$(X=0, Y=0)$	$(X=0, Y=0)$	$(X=0, Y=1)$	$(X=0, Y=1)$	$(X=0, Y=1)$
4	$(X=1, Y=1)$	$(X=0, Y=1)$	$(X=0, Y=1)$	$(X=0, Y=2)$	$(X=0, Y=2)$	$(X=0, Y=2)$
5	$(X=1, Y=1)$	$(X=0, Y=1)$	$(X=0, Y=1)$	$(X=0, Y=2)$	$(X=0, Y=2)$	$(X=0, Y=2)$
6	$(X=1, Y=1)$	$(X=0, Y=1)$	$(X=0, Y=1)$	$(X=0, Y=2)$	$(X=0, Y=2)$	$(X=0, Y=2)$

and by counting, the joint pmf of (X, Y) is given in the table below:

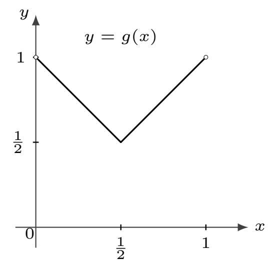
Y \ X	0	1	2
0	$\frac{4}{36}$	$\frac{4}{36}$	$\frac{1}{36}$
1	$\frac{12}{36}$	$\frac{6}{36}$	$\times$
2	$\frac{9}{36}$	$\times$	$\times$

with $P(X = i, Y = j) = 0$ otherwise.

2. [Exam02] (32pts) Species A and Species B exclusively compete in a region for control of a limited amount of a certain resource. Let the random variable X be the *proportion* of this resource controlled by Species A where $X \sim \text{Uniform}(0, 1)$, then the proportion of the resource controlled by Species B is $1–X$. Let $g(x) = \max(x, 1 - x)$,

then $Y = g(X)$ is the proportion of this resource controlled by the dominant species, that is

$$Y = g(X) = \begin{cases} 1 - X, & \text{if } 0 < X < \frac{1}{2}, \\ X, & \text{if } \frac{1}{2} \le X < 1. \end{cases}$$



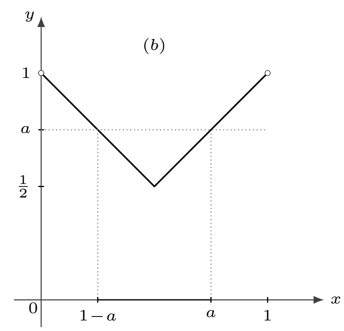
(a) (8pts) Find the probability that $|X - 0.7| > 0.1$.

(b) (8pts) Find $E[g(X)]$, the proportion of this resource expected to be controlled by the dominant species.

(c) (8pts) Find the *cumulative distrubution function* $F_Y(a)$ of $Y = \max(X, 1 - X)$. Define the cdf for all $\mathbb{R}$.

(d) (8pts) What is the probability that the dominant species will control more than 70% of the resource?

Solution:



(a)(8pts) Note that $P(|X - 0.7| > 0.1) = 1 - P(|X - 0.7| \le 0.1)$ where

$$|X - 0.7| \le 0.1 \;\Rightarrow\; -0.1 \le X - 0.7 \le 0.1 \;\Rightarrow\; 0.6 \le X \le 0.8$$

so $P(|X - 0.7| > 0.1) = 1 - P(0.6 \le X \le 0.8) = 1 - 0.2 = 0.8$.

(b)(8pts) We need to find $E[g(X)]$. Keeping in mind that $f(x) = \begin{cases} 1, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$ then, by a theorem in the textbook, we claim that

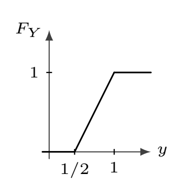
$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx = \int_0^{1/2} (1 - x)\, dx + \int_{1/2}^{1} x\, dx$$
$$= \left(x - \frac{x^2}{2}\right)\Big|_0^{1/2} + \frac{x^2}{2}\Big|_{1/2}^{1} = \frac{3}{8} + \frac{3}{8} = \frac{3}{4}.$$

(c)(8pts) Since $Y \in [\frac{1}{2}, 1)$, if $a < \frac{1}{2}$ then $P(Y \le a) = 0$ and if $a \ge 1$ then $P(Y \le a) = 1$, finally, if $a \in [\frac{1}{2}, 1)$ then, since $\max(x, 1 - x) \le a$ if and only if $x \le a$ and $1 - x \le a$ (see graph), we have

$$P(Y \le a) = P\big(\{1 - X \le a\} \cap \{X \le a\}\big) = P\big(1 - a \le X \le a\}\big) = \int_{1-a}^{a} 1\, dx = a - (1 - a) = 2a - 1,$$

thus, the cdf of Y is

$$F_Y(a) = \begin{cases} 0, & \text{if } a < \frac{1}{2}, \\ 2a - 1, & \text{if } \frac{1}{2} \le a < 1, \\ 1, & \text{if } a \ge 1. \end{cases}$$



$\boxed{\text{Alternate solution}}$ If $a \in [\frac{1}{2}, 1)$ then, from the given graph and due to the symmetry, we see that $y = g(x)$ is the function $|x|$ shifted $\frac{1}{2}$ units to the right and $\frac{1}{2}$ units up, that is, $g(x) = |x - \frac{1}{2}| + \frac{1}{2}$ and $Y = |X - \frac{1}{2}| + \frac{1}{2}$, so

$$
\begin{aligned}
P(Y \le a) = P\big(|X - \tfrac{1}{2}| + \tfrac{1}{2} \le a\big) &= P\big(|X - \tfrac{1}{2}| \le a - \tfrac{1}{2}\big) \\
&= P\big(-a + \tfrac{1}{2} \le X - \tfrac{1}{2} \le a - \tfrac{1}{2}\big) \\
&= P\big(1 - a \le X \le a\}\big) = \int_{1-a}^{a} 1\, dx = a - (1 - a) = 2a - 1.
\end{aligned}
$$

(d)(8pts) Using the cdf of Y, the probability that the dominant species will control more than 70% of the resource is

$$P(Y > 0.7) = 1 - P(Y \le 0.7) = 1 - F_Y(0.7) = 1 - \big(2{\cdot}0.7 - 1\big) = 1 - 0.4 = 0.6.$$

3. [EXAM02] (28pts) Let D be the region bounded by $x = 0$, $y = 0$ and $y = 2 - 2x$. Let (X, Y) have the joint probability density function

$$f(x, y) = \begin{cases} \dfrac{3}{4}(y + 2x), & \text{for } x \text{ and } y \text{ in region } D, \\ 0, & \text{otherwise.} \end{cases}$$

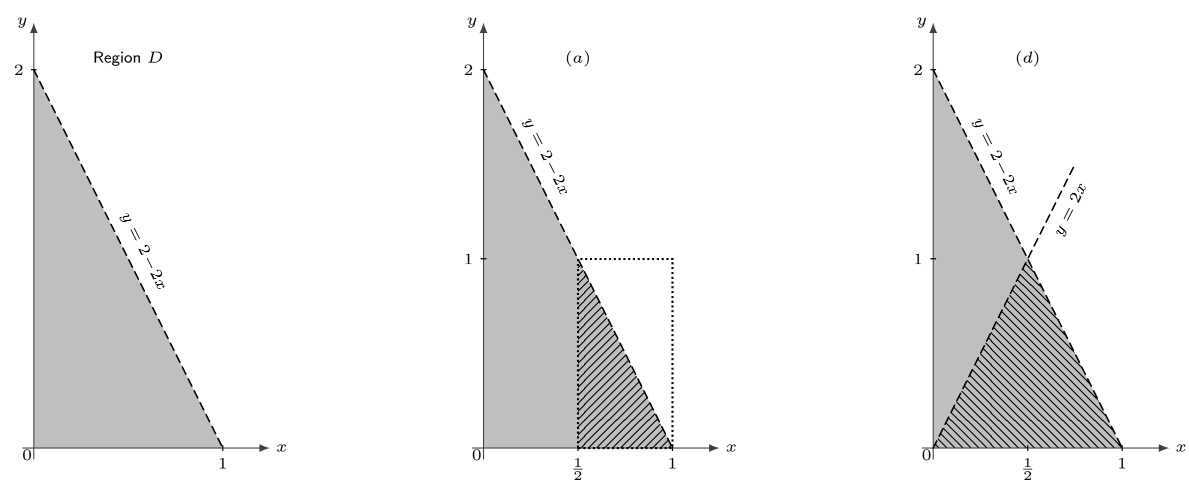
(a) (7pts) Set-up, but *do not solve* an integral (or integrals) to find $P(\frac{1}{2} < X < 1,\ Y < 1)$.

(b) (7pts) Find the *marginal probability density function* $f_X(x)$ of X. Define the pdf for all $\mathbb{R}$.

(c) (7pts) Find the expectation $E[X]$.

(d) (7pts) Set-up, but *do not solve* an integral (or integrals) to find $P(Y < 2X)$.

Solution:



(a) (7pts) First note that $\{\frac{1}{2} < X < 1,\ Y < 1\} \cap \operatorname{supp}(f) = \{\frac{1}{2} < X < 1,\ 0 < Y < 2 - 2X\}$ so

$$P(\tfrac{1}{2} < X < 1,\ 0 < Y < 1) = P(\tfrac{1}{2} < X < 1,\ 0 < Y < 2 - 2X) = \int_{\frac{1}{2}}^{1} \int_{0}^{2-2x} \frac{3}{4}(y + 2x)\, dy dx.$$

Alternate: Note that $P(\frac{1}{2} < X < 1,\ Y < 1) = P(\frac{1}{2} < X < 1 - \frac{Y}{2},\ 0 < Y < 1)$.

(b) (7pts) For each $x \in (0, 1)$, we have

$$
\begin{aligned}
f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \int_{0}^{2-2x} \frac{3}{4}(y + 2x)\, dy \\
= \frac{3}{4}\Big(\frac{y^2}{2} + 2xy\Big)\Big|_0^{2-2x} = \frac{3}{4}\Big(\frac{4 - 8x + 4x^2}{2} + 4x - 4x^2\Big) = \frac{3}{4}(2 - 2x^2)
\end{aligned}
$$
and 0 otherwise.

(c) (7pts) Using the pdf of X, we have

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \frac{3}{4}\int_0^1 x(2-2x^2)\, dx = \frac{3}{2}\int_0^1 (x-x^3)\, dx = \frac{3}{2}\Big(\frac{x^2}{2}-\frac{x^4}{4}\Big)\Big|_0^1 = \frac{3}{2}\cdot\frac{1}{4} = \frac{3}{8}.$$

(d) (7pts) Note that the intersection point of $y = 2x$ and $y = 2-2x$ is $(\frac{1}{2}, 1)$, thus

$$\begin{aligned} P(Y < 2X) &= \int_0^{\frac{1}{2}}\int_0^{2x} \frac{3}{4}(y+2x)\, dydx + \int_{\frac{1}{2}}^{1}\int_0^{2-2x} \frac{3}{4}(y+2x)\, dydx \\ &= \int_0^1 \int_{\frac{y}{2}}^{1-\frac{y}{2}} \frac{3}{4}(y+2x)\, dxdy. \end{aligned}$$