Write your name below. This exam is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You are allowed to use two pages of notes (one piece of paper, front and back). You are not allowed to use a calculator or any computational software.

Name: Section:
(Chi/9:05/001; Grooms/11:15/002; Grooms/1:25/003)

1. (28 points: 4 each) If the statement is **always true** mark "TRUE"; if it is possible for the statement to be false then mark "FALSE". **No justification is necessary**.

_____(a) A matrix whose columns form an orthogonal basis of $\mathbb{R}^n$ is an orthogonal matrix.

_____(b) An orthogonal matrix has determinant 1.

_____(c) When obtaining a QR decomposition for a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ with the Householder method, the resulting $\mathbf{Q}$ matrix will be rectangular in the case that $m > n$.

_____(d) Suppose that $\mathbf{A}$ is symmetric and has an LU factorization without pivoting. True or False: $\mathbf{A}$ is positive definite.

_____(e) If the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are mutually orthogonal and nonzero, then they are linearly independent. (Mutually orthogonal means $\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = 0$ whenever $i \neq j$.)

_____(f) Let $\boldsymbol{u}$ be a unit vector. True or False: $(\mathbf{I} - \boldsymbol{u}\boldsymbol{u}^T)\boldsymbol{x}$ is the orthogonal projection of $\boldsymbol{x}$ onto $\operatorname{span}\{\boldsymbol{u}\}^\perp$.

_____(g) Let $p(\boldsymbol{x}) = \boldsymbol{x}^T\mathbf{K}\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{f} + c$, and let $\mathbf{K}$ be symmetric and non-negative definite. True or False: If $\boldsymbol{f}$ is orthogonal to the kernel of $\mathbf{K}$ then the function p has a minimum value.

2. (30 points, 6 each) Let $\mathcal{W} \subset \mathbb{R}^3$ be the plane spanned by the vectors

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

and let $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. We want to find the $\mathbf{w} \in \operatorname{span}(\mathbf{w}_1, \mathbf{w}_2)$ that is closest to $\mathbf{b}$. We do this by finding the $\mathbf{w}$ that minimizes the distance between $\mathbf{b}$ and all the vectors in $\operatorname{span}(\mathbf{w}_1, \mathbf{w}_2)$. We denote the solution as $\mathbf{w}^\star$.

(a) Write this problem in terms of minimizing a quadratic function $p(\mathbf{x}) = \mathbf{x}^T\mathbf{K}\mathbf{x} - 2\mathbf{x}^T\mathbf{f} + c$. Assume that we employ the standard dot product so that we are minimizing the Euclidean distance between $\mathbf{b}$ and $\mathbf{w}$. What are $\mathbf{K}$, $\mathbf{f}$, and c?

(b) What are the coordinates of $\mathbf{w}^\star$ in terms of $\mathbf{w}_1$ and $\mathbf{w}_2$?

(c) What is this closest point $\mathbf{w}^\star$?

(d) What is the *squared* Euclidean distance between $\mathbf{w}^\star$ and $\mathbf{b}$? (Hint: No need to take square roots!)

(e) Suppose that we employ instead the weighted norm defined by $\|\mathbf{v}\|^2 = 3v_1^2 + v_2^2 + \frac{1}{3}v_3^2$. What are $\mathbf{K}$ and $\mathbf{f}$ in this case?

3. (30 points, 10 each) Consider the function

$$p(x, y, z) = x^2 - 4xy + 6xz + 8x + 8y^2 - 20yz - 24y + 14z^2 + 34z + 5.$$

(a) This function can be written as $p(x, y, z) = \boldsymbol{x}^T \mathbf{K} \boldsymbol{x} - 2\boldsymbol{x}^T \boldsymbol{f} + c$, where $\boldsymbol{x} = (x, y, z)^T$. What are $\mathbf{K}$, $\boldsymbol{f}$, and c?

(b) Find all critical points of this function.

(c) For each critical point, say whether it corresponds to a (local) minimum, maximum, or saddle point. Explain your reasoning.

4. (12 points) Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and let $\mathbf{Q}$ be an $n \times n$ orthogonal matrix. Prove that the (Euclidean) angle between $\boldsymbol{x}$ and $\boldsymbol{y}$ is the same as the (Euclidean) angle between $\mathbf{Q}\boldsymbol{x}$ and $\mathbf{Q}\boldsymbol{y}$.

5. **Bonus**

   (i) (6 points, 2 each) Prove that the Householder reflection matrix $\mathbf{H} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$, where $\mathbf{u}$ is a unit vector, is (a) its own inverse, (b) symmetric, and (c) orthogonal.

   (ii) (2 points) Prove that orthogonal matrices preserve length. (You can use Euclidean length.)

   (iii) (2 points) Prove that the inverse of an orthogonal matrix is also orthogonal.

