1. [2350/112024 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

(a) The vector field $\mathbf{V}(x,y,z) = (z-x)\,\mathbf{i} + (z-y)\,\mathbf{j} + 2z\,\mathbf{k}$ is both incompressible and irrotational.

(b) If $x = \dfrac{u}{v}, y = \dfrac{v}{w}, z = \dfrac{w}{u}$, then $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = 0$.

(c) $\displaystyle\int_{-1}^{0}\int_{0}^{1+y} e^{x+y}\,\mathrm{d}x\,\mathrm{d}y + \int_{0}^{1}\int_{0}^{1-y} e^{x+y}\,\mathrm{d}x\,\mathrm{d}y = \int_{0}^{1}\int_{x-1}^{1-x} e^{x+y}\,\mathrm{d}y\,\mathrm{d}x$

(d) If $\mathcal{C}$ is the curve $y = x^2, 0 \le x \le 1$, then the area under the graph of $f(x,y) = x + 2y$ above $\mathcal{C}$ is $\displaystyle\iint_{\mathcal{C}} (x+2y)\,\mathrm{d}S$.

(e) For any sufficiently differentiable vector field $\mathbf{F}$, $\nabla \times (\nabla \cdot \mathbf{F})$ is a meaningful expression.

SOLUTION:

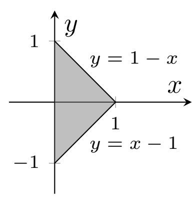
(a) **FALSE**

$$\nabla \cdot \mathbf{V} = 0 \text{ (incompressible)} \qquad \text{and} \qquad \nabla \times \mathbf{V} = -\mathbf{i} + \mathbf{j} \ne \mathbf{0} \text{ (not irrotational)}$$

(b) **TRUE**

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1/v & -u/v^2 & 0 \\ 0 & 1/w & -v/w^2 \\ -w/u^2 & 0 & 1/u \end{vmatrix} = \frac{1}{uvw} - \frac{uvw}{u^2v^2w^2} = \frac{1}{uvw} - \frac{1}{uvw} = 0$$

(c) **TRUE** The region of integration is shown.



(d) **FALSE** The area is $\displaystyle\int_{\mathcal{C}} (x+2y)\,\mathrm{d}s$, a line integral.

(e) **FALSE** $\nabla \cdot \mathbf{F}$ is a scalar.

2. [2350/112024 (20 pts)] On a certain day the density of lightning strikes in the Colorado Rocky Mountains was found to be well-approximated by the function

$$\delta(x,y) = \left(x^2+y^2\right)^{3/2}\left[1 + 12\tan^{-1}(y/x)\right]$$

strikes per square mile. Some colleagues of yours need to calculate the total number of lightning strikes, N, in a certain area that day to help determine the risk of new forest fires. They have decided that this can be accomplished by computing the value of

$$N = \int_{\sqrt{3}/2}^{1}\int_{\sqrt{1-x^2}}^{x/\sqrt{3}} \delta(x,y)\,\mathrm{d}y\,\mathrm{d}x + \int_{1}^{\sqrt{3}}\int_{0}^{x/\sqrt{3}} \delta(x,y)\,\mathrm{d}y\,\mathrm{d}x + \int_{\sqrt{3}}^{2}\int_{0}^{\sqrt{4-x^2}} \delta(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

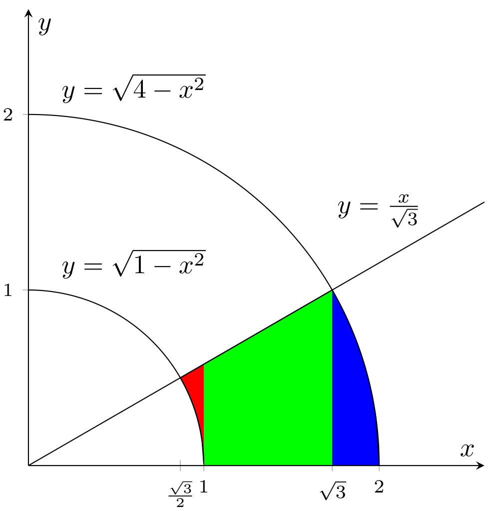
However, since they took Calculus 3 many moons ago, they cannot seem to recall how to do this calculation. Come to their assistance by determining the total number lightning strikes in their area of interest, simplifying your final answer. Hint: Draw the region of integration and consider using another coordinate system.

SOLUTION:

The form of the density function itself strongly suggests the use of polar coordinates, giving

$$\delta(r,\theta) = r^3(1+12\theta) \qquad \left(\text{note that } \theta = \tan^{-1}\frac{y}{x} \text{ in the first quadrant}\right)$$

Using the bounds on the integrals we can draw the following figure, which also suggests using polar coordinates:



From the figure:

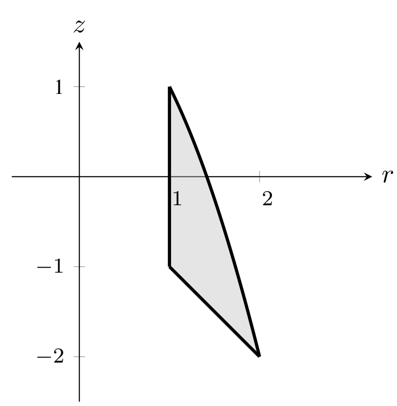
$$\begin{aligned}
&y=\sqrt{1-x^2} \implies x^2+y^2=1 \implies r=1\\
&y=\sqrt{4-x^2} \implies x^2+y^2=4 \implies r=2\\
&y=x/\sqrt{3} \implies y/x=1/\sqrt{3} \implies \tan^{-1}(y/x)=\tan^{-1}(1/\sqrt{3}) \implies \theta=\pi/6
\end{aligned}$$

so the bounds of integration in polar coordinates are $1 \le r \le 2$ and $0 \le \theta \le \pi/6$. Thus the total number of lightning strikes becomes

$$\begin{aligned}
N &= \int_0^{\pi/6}\int_1^2 r^3(1+12\theta)r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\pi/6}\int_1^2 r^4(1+12\theta)\,\mathrm{d}r\,\mathrm{d}\theta\\
&= \int_0^{\pi/6}(1+12\theta)\frac{1}{5}r^5\Big|_1^2\,\mathrm{d}\theta = \frac{31}{5}\int_0^{\pi/6}(1+12\theta)\,\mathrm{d}\theta = \frac{31}{5}\left(\theta+6\theta^2\right)\Big|_0^{\pi/6}\\
&= \frac{31}{5}\left[\frac{\pi}{6}+6\left(\frac{\pi^2}{36}\right)\right] = \frac{31\pi}{30}(1+\pi)\text{ lightning strikes}
\end{aligned}$$

■

3. [2350/112024 (20 pts)] The portion of a solid metal machine part is shown in a constant θ-plane in the figure below. The curved portion (on the right) is given by $x^2+y^2+z=2$. The other boundaries are straight lines. If the weight density of the metal is $\delta(x,y,z)=\dfrac{6y}{x^2+y^2}$ pounds per cubic inch, find the weight of that portion of the metal part lying between the planes $y=x$ and $y=-x$ with $y \ge 0$.



SOLUTION:

Let $\mathcal{W}$ be the solid region of interest and use cylindrical coordinates so that $\delta(r,\theta,z) = \dfrac{6r\sin\theta}{r^2} = \dfrac{6\sin\theta}{r}$.

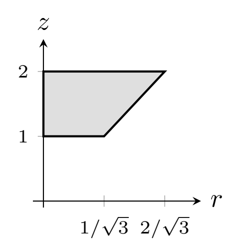
$$\begin{aligned}
\text{Weight} = \iiint_{\mathcal{W}} \delta(x,y,z)\,\mathrm{d}V &= \int_{\pi/4}^{3\pi/4}\int_1^2\int_{-r}^{2-r^2} 6\sin\theta\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \left(\int_{\pi/4}^{3\pi/4}\sin\theta\,\mathrm{d}\theta\right)\left[6\int_1^2 \left(2-r^2+r\right)\mathrm{d}r\right] \\
&= 6\left(\cos\theta\Big|_{3\pi/4}^{\pi/4}\right)\left(2r-\frac{r^3}{3}+\frac{r^2}{2}\right)\Big|_1^2 \\
&= 6\sqrt{2}\left[\left(4-\frac{8}{3}+2\right)-\left(2-\frac{1}{3}+\frac{1}{2}\right)\right] \\
&= 6\sqrt{2}\left(\frac{7}{2}-\frac{7}{3}\right) = 7\sqrt{2}\text{ pounds}
\end{aligned}$$

■

4. [2350/112024 (8 pts)] Convert $\displaystyle\int_1^2\int_0^\pi\int_0^{z/\sqrt{3}} r^3\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$ to spherical coordinates using the order $\mathrm{d}\theta\,\mathrm{d}\rho\,\mathrm{d}\phi$. **Do not** evaluate either integral.

SOLUTION:

The figure shows the region of integration in the rz-plane.



$$\int_0^{\pi/6}\int_{\sec\phi}^{2\sec\phi}\int_0^{\pi} \rho^4\sin^3\phi\,\mathrm{d}\theta\,\mathrm{d}\rho\,\mathrm{d}\phi$$

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5. [2350/112024 (22 pts)] In the written homework, you were asked to find the volume of the solid region bounded by $f(x,y) = 4-x^2-2y^2$ and the xy-plane. This resulted in a fairly lengthy calculation. We will do this problem here in a much easier way.

(a) [6 pts] Find $\mathcal{D}$, the region in the xy-plane over which you will integrate to compute the volume.

(b) [10 pts] Use the change of variables $u = x, v = \sqrt{2}y$ to write a new integral that will give the volume, but **do not evaluate** it.

(c) [6 pts] Use another (familiar) change of variables to find an integral that computes the volume and **evaluate** it.

SOLUTION:

(a) $\mathcal{D}$ is the elliptical disk $x^2+2y^2 \le 4$.

(b)

$$u = x, v = \sqrt{2}y \implies x = u, y = v/\sqrt{2}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{\sqrt{2}}$$

$$x^2+2y^2 \le 4 \to u^2+v^2 \le 4$$

$$\text{Volume} = \frac{1}{\sqrt{2}}\int_{-2}^{2}\int_{-\sqrt{4-u^2}}^{\sqrt{4-u^2}} \left(4-u^2-v^2\right)\,\mathrm{d}v\,\mathrm{d}u = \frac{1}{\sqrt{2}}\int_{-2}^{2}\int_{-\sqrt{4-v^2}}^{\sqrt{4-v^2}} \left(4-u^2-v^2\right)\,\mathrm{d}u\,\mathrm{d}v$$

(c) Now switch to polar coordinates.

$$\text{Volume} = \frac{1}{\sqrt{2}} \int_0^{2\pi} \int_0^2 \left(4 - r^2\right) r \, \mathrm{d}r \, \mathrm{d}\theta = \frac{2\pi}{\sqrt{2}} \left(2r^2 - \frac{r^4}{4}\right)\Bigg|_0^2 = \frac{8\pi}{\sqrt{2}} = 4\sqrt{2}\pi$$

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6. [2350/112024 (20 pts)] The metal roof of a storage shed is described by the function $3z = 1 - x^2,\ z \geq 0,\ 0 \leq y \leq 2$. A wire runs along the edge of the roof where $y = 2$. In both of the following parts, you will set up an integral that computes the given quantity. Your final answer will be an integral simplified completely so that the only remaining step is to actually perform the integration, which you will **not** do.

   (a) [10 pts] The total charge, Q_r, on the roof if the charge density is $q(x, y, z) = xyz$ coulombs per square meter.

   (b) [10 pts] The total mass, m_w, of the wire if the mass density is $\rho(x, y, z) = 3xyz$ kilograms per meter.

SOLUTION:

(a) $Q_r = \iint_{\mathcal{S}} xyz \, \mathrm{d}S$, where $\mathcal{S}$ is the roof of the shed, given by $g(x, y, z) = 3z + x^2 = 1$. Then, if we project onto the xy-plane,

$$\mathcal{R} : -1 \leq x \leq 1, 0 \leq y \leq 2,\ \mathbf{p} = \mathbf{k}$$

$$\nabla g = \langle 2x, 0, 3 \rangle$$

$$\|\nabla g\| = \sqrt{4x^2 + 9}$$

$$|\nabla g \cdot \mathbf{p}| = 3$$

$$q(x, y, z) \to \frac{1}{3} xy \left(1 - x^2\right)$$

$$Q_r = \frac{1}{9} \int_{-1}^{1} \int_0^2 xy \left(1 - x^2\right) \sqrt{4x^2 + 9} \, \mathrm{d}y \, \mathrm{d}x$$

(b) $m_w = \int_{\mathcal{C}} 3xyz \, \mathrm{d}s$, where $\mathcal{C}$ is the wire, which can be parameterized as $\mathbf{r}(t) = \left\langle t, 2, \frac{1 - t^2}{3} \right\rangle, -1 \leq t \leq 1$. Then

$$\mathbf{r}'(t) = \left\langle 1, 0, -\frac{2t}{3} \right\rangle \implies \|\mathbf{r}'(t)\| = \sqrt{1 + \frac{4t^2}{9}} = \frac{1}{3}\sqrt{4t^2 + 9}$$

$$\rho(\mathbf{r}(t)) = 2t \left(1 - t^2\right)$$

$$m_w = \frac{1}{3} \int_{-1}^{1} 2t(1 - t^2)\sqrt{4t^2 + 9} \, \mathrm{d}t$$

■