Write your name below. This exam is worth 100 points. You must show all your work to receive credit on each problem and must fully simplify your answers unless otherwise instructed. You are allowed to use one page of notes. You are not allowed to use a calculator or any computational software.

Name:

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1. (x points) Let $f(x, y) = \ln(9 - x^2 - 4y^2)$

   (a) (x points) Sketch the domain of f, labeling all relevant points.

   Solution: Our domain is the region of the plane where the argument of the logarithm is positive:

$$\begin{aligned} 9 - x^2 - 4y^2 &> 0 \\ x^2 + 4y^2 &< 9 \\ \frac{x^2}{9} + \frac{4y^2}{9} &< 1 \end{aligned}$$

   The domain is the interior of an ellipse centered at the origin:

   (b) (x points) Find a nonzero vector that is orthogonal to the level curve of f passing through the point $(2, 1)$.

   Solution: The gradient vector at $(1, 0)$ is orthogonal to the level curve through that point, so we have

$$\begin{aligned} \nabla f(x, y) &= \langle f_x, f_y \rangle \\ \nabla f(x, y) &= \langle \frac{-2x}{9 - x^2 - 4y^2}, \frac{-8y}{9 - x^2 - 4y^2} \rangle \\ \nabla f(x, y) &= \frac{1}{9 - x^2 - 4y^2} \langle -2x, -8y \rangle \\ \nabla f(2, 1) &= \frac{1}{9 - 4 - 4} \langle -4, -8 \rangle \\ &= \langle -4, -8 \rangle \end{aligned}$$

   (c) (x points) Find a vector pointing along the level curve at the same point.

   Solution: The tangent vector of a level curve is given by $\langle f_y, -f_x \rangle$, so we have

$$\mathbf{v} = \langle -8, 4 \rangle$$

   (d) (x points) What is the rate of change of $f(x, y)$ at the point $(1, 1)$ in the direction of the vector $4\mathbf{i} + 5\mathbf{j}$?

   Solution: We first must find the unit vector along $\mathbf{v}$:

$$\begin{aligned} \mathbf{u} &= \frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \frac{1}{\sqrt{41}} \langle 4, 5 \rangle \end{aligned}$$

We then find the directional derivative in the direction of **u**:

$$\begin{aligned}
D_{\mathbf{u}} &= \nabla f(1,1) \cdot \mathbf{u} \\
&= \frac{1}{9-1-4}\langle -2, -8\rangle \cdot \frac{1}{\sqrt{41}}\langle 4, 5\rangle \\
&= \frac{1}{4\sqrt{41}}\left(-8-40\right) \\
\\
&= -\frac{12}{\sqrt{41}}
\end{aligned}$$

(e) (x points) Let $x(s,t) = 3s\cos(t)$ and $y(s,t) = \frac{3}{2}s\sin(t)$. Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

Solution:

$$\begin{aligned}
\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\
\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}
\end{aligned}$$

We'll need the partial derivatives with respect to x and y in terms of s and t

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{-2x}{9-x^2-4y^2} \\
&= \frac{-6s\cos(t)}{9-9s^2\cos^2 t - 9s^2\sin^2(t)} \\
&= \frac{-2s\cos(t)}{3(1-s^2)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{-8y}{9-x^2-4y^2} \\
&= \frac{-12s\sin(t)}{9-9s^2\cos^2 t - 9s^2\sin^2(t)} \\
&= \frac{-4s\sin(t)}{3(1-s^2)}
\end{aligned}$$

The partial derivatives of x and y are

$$\begin{aligned}
\frac{\partial x}{\partial s} &= 3\cos(t) \\
\frac{\partial x}{\partial t} &= -3\sin(t) \\
\frac{\partial y}{\partial s} &= \frac{3}{2}\sin(t) \\
\frac{\partial y}{\partial t} &= \frac{3}{2}s\cos(t)
\end{aligned}$$

Bringing this together

$$\begin{aligned}
\frac{\partial f}{\partial s} &= \frac{-2s\cos(t)}{3(1-s^2)}(3\cos(t)) + \frac{-4s\sin(t)}{3(1-s^2)}\frac{3}{2}\sin(t) \\
&= \frac{-2s\cos^2(t)}{(1-s^2)} + \frac{-2s\sin^2(t)}{(1-s^2)} \\
&= \frac{-2s}{(1-s^2)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{-2s\cos(t)}{3(1-s^2)}(-3s\sin(t)) + \frac{-4s\sin(t)}{3(1-s^2)}\frac{3}{2}s\cos(t) \\
&= \frac{2s^2\cos(t)\sin(t)}{(1-s^2)} + \frac{-2s^2\sin(t)\cos(t)}{(1-s^2)} \\
&= 0
\end{aligned}$$

2. (x points) Let $f(x, y) = 3xy - x^2y - xy^2$.

(a) (x points) Find all the critical points of $f(x, y)$.

Solution: We set both first partial derivatives equal to 0 and solve:

$$\begin{aligned} f_x &= 3y - 2xy - y^2 \\ &= (3 - 2x - y)y \end{aligned} \qquad\qquad \begin{aligned} f_y &= 3x - x^2 - 2xy \\ &= (3 - x - 2y)x \end{aligned}$$

Setting these equal to 0 and solving gives:

$$y = 3 - 2x \tag{1}$$
$$y = 0 \tag{2}$$
$$x = 3 - 2y \tag{3}$$
$$x = 0 \tag{4}$$

So our first critical point is the origin $(0, 0)$. We find two more by substituting $x = 0$ and $y = 0$ into our line equations, getting $(0, 3)$ and $(3, 0)$. Our final critical point is the intersection of the two lines. Substituting (3) into (1) gives

$$\begin{aligned} y &= 3 - 2(3 - 2y) \\ y &= 3 - 6 + 4y \\ -3y &= -3 \\ y &= 1 \end{aligned}$$

Substituting into (3) gives $x = 1$ so our final critical point is $(1, 1)$.

(b) (x points) Classify the critical points as local maxima, local minima, or saddle points using the second derivative test.

Solution: Our second derivatives are

$$\begin{aligned} f_{xx} &= -2y \\ f_{xy} &= 3 - 2x - 2y \\ f_{yy} &= -2x \end{aligned}$$

We calculate the discriminant D at each point:

$$\begin{aligned} D &= f_{xx}f_{yy} - (f_{xy})^2 \\ D &= (-2y)(-2x) - (3 - 2x - 2y)^2 \\ (0, 0) : D &= (0)(0) - (3)^2 = -9 \\ (0, 3) : D &= (-6)(0) - (-3)^2 = -9 \\ (3, 0) : D &= (0)(-6) - (-3)^2 = -9 \\ (1, 1) : D &= (-2)(-2) - (-1)^2 = 3 \end{aligned}$$

We can immediately say that there are saddle points at $(0, 0)$, $(0, 3)$, and $(3, 0)$ and that $(1, 1)$ is the location of a local extrema. Since $f_{xx}(1, 1) = -2$, it is a local maximum.

(c) (x points) Let T be the triangular region bounded by

$$\begin{aligned} x &= 0 \\ y &= 0 \\ y &= 2 - x \end{aligned}$$

Find the extreme values of $f(x, y)$ on T.

Solution: Our region is the triangle with corners $(0, 0)$, $(0, 2)$, and $(2, 0)$

Along $x = 0$ we have $f(0, y) = 0$. The function is 0 everywhere on the y-axis. Similarly, along $y = 0$ we have $f(x, 0) = 0$, so the function is also 0 along the x-axis.

Finally, along $y = 2 - x$ we find that

$$\begin{aligned} f(x, 2-x) &= 3x(2-x) - x^2(2-x) - x(2-x)^2 \\ &= 6x - 3x^2 - 2x^2 + x^3 - 4x + 4x^2 - x^3 \\ &= 2x - x^2 \end{aligned}$$

This is a quadratic with an extreme value located at $x = -\dfrac{2}{2(-1)} = 1$ so we have the extremum on the boundary:

$$f(1, 1) = 1$$

This is the local maximum we found earlier, so the extreme values of f on T are 0 and 1..

3. (x points) Find the following limits or show that they do not exist.

   (a) (x points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2(y)}{x^4+2y^4}$

   Solution: This limit does not exist. Along $x=0$ we have

$$\lim_{(x,y)\to(0,0)} \frac{0^2\sin^2(y)}{0^4+2y^4} = 0$$

   but along $y=x$ we have

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2(x)}{x^4+2x^4} &= \lim_{(x,y)\to(0,0)} \frac{\sin^2(x)}{3x^2}\\ &= \frac{1}{3}\end{aligned}$$

   As these two limits are not equal, the limit does not exist.

   (b) (x points) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x-y}{\sqrt{x}-\sqrt{y}}$

   Solution: Here we can factor and cancel:

$$\begin{aligned}\lim_{(x,y)\to(1,1)} \frac{x-y}{\sqrt{x}-\sqrt{y}} &= \lim_{(x,y)\to(1,1)} \frac{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}{\sqrt{x}-\sqrt{y}}\\ &= \lim_{(x,y)\to(1,1)} \sqrt{x}+\sqrt{y}\\ &= 2\end{aligned}$$

   (c) (x points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^5+y^3}$

   Solution: This limit does not exist. Along $y=0$ we have

$$\lim_{(x,y)\to(0,0)} \frac{x^3\cdot 0}{x^5+0^3} = 0$$

   but along $y=x^2$ we have

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^3\cdot x^2}{x^5+x^6} &= \lim_{(x,y)\to(0,0)} \frac{x^5}{x^5(1+x)}\\ &= \lim_{(x,y)\to(0,0)} \frac{1}{(1+x)}\\ &= 1\end{aligned}$$

   As these two limits are not equal, the limit does not exist.

4. (x points) A long straight railroad tunnel has a cross section in the shape of a gaussian: $y = e^{-x^2}$. Use the method of Lagrange Multipliers to find the width and height of the largest rectangular cross-section railroad cars that can fit through the tunnel.

Solution:

We want to maximize the cross-sectional area subject to the constraint $y = e^{-x^2}$. The area is given by $f(x, y) = 2xy$ while our constraint equation is $g(x, y) = y - e^{-x^2} = 0$.

$$\begin{aligned} \nabla f &= \langle 2y, 2x \rangle \\ \nabla g &= \langle 2xe^{-x^2}, 1 \rangle \\ \nabla f &= \lambda \nabla g \\ \langle 2y, 2x \rangle &= \lambda \langle 2xe^{-x^2}, 1 \rangle \end{aligned}$$

So we have two equations

$$\begin{aligned} 2y &= 2\lambda xe^{-x^2} \\ 2x &= \lambda \end{aligned}$$

Substituting $\lambda = 2x$ into the first equation gives

$$\begin{aligned} 2y &= 2(2x)xe^{-x^2} \\ y &= 2x^2e^{-x^2} \end{aligned}$$

Substituting this into our constraint gives

$$\begin{aligned} y - e^{-x^2} &= 0 \\ 2x^2e^{-x^2} - e^{-x^2} &= 0 \\ 2x^2 - 1 &= 0 \\ x &= \frac{1}{\sqrt{2}} \end{aligned}$$

Substitute this into our y equation to get

$$y = e^{-1/2}$$

So the cars are $\dfrac{2}{\sqrt{2}}$ wide and $e^{-1/2}$ tall.

5. (x points) Let $f(x, y) = e^{xy}$

(a) (x points) Find the tangent plane to the function near the point $(1, 2)$.
Solution: Near $(1, 2)$, the tangent plane is given by

$$T(x, y) = f(1, 2) + f_x(1, 2) \cdot (x - 1) + f_y(1, 2) \cdot (y - 2)$$

The partial derivatives are given by

$$\begin{aligned} f_x &= ye^{xy} & \qquad f_x(1,2) &= 2e^2 \\ f_y &= xe^{xy} & \qquad f_y(1,2) &= e^2 \end{aligned}$$

So we have

$$\begin{aligned} T(x, y) &= e^2 + (2e^2)(x - 1) + (e^2)(y - 2) \\ T(x, y) &= -3e^2 + 2e^2 x + e^2 y \\ T(x, y) &= (2x + y - 3)e^2 \end{aligned}$$

(b) (x points) Find the degree 2 Taylor approximation of $f(x, y)$ at the same point. You may leave your answer in factored form.
Solution: To find the degree 2 approximation, we must add

$$\frac{1}{2}\left(f_{xx}(1, 2)(x - 1)^2 + 2f_{xy}(1, 2)(x - 1)(y - 2) + f_{yy}(1, 2)(y - 2)^2\right)$$

to our tangent plane. Finding all the second partial derivatives:

$$\begin{aligned} f_{xx} &= y^2 e^{xy} & \qquad f_{xx}(1,2) &= 4e^2 \\ f_{xy} &= (1 + xy)e^{xy} & \qquad f_{xy}(1,2) &= 3e^2 \\ f_{yy} &= x^2 e^{xy} & \qquad f_{yy}(1,2) &= e^2 \end{aligned}$$

So we have

$$Q(x, y) = T(x, y) + \frac{1}{2}\left(4e^2(x - 1)^2 + 6e^2(x - 1)(y - 2) + e^2(y - 2)^2\right)$$

(c) (x points) What is the maximum error of the degree 1 Taylor approximation when $|x - 1| < 0.1$ and $|y - 2| < 0.1$?

Solution: We first calculate the maximum values of our second derivatives on the region. Since they are all increasing functions in both variables, they all attain their maximums at the point $(1.1, 2.1)$:

$$\begin{aligned} f_{xx}(1.1, 2.1) &= 4.41e^{2.31} \\ f_{xy}(1.1, 2.1) &= 3.31e^{2.31} \\ f_{yy}(1.1, 2.1) &= 1.21e^{2.31} \end{aligned}$$

So we have a maximum value of $4.41e^{2.31}$ and so our error is bounded by

$$\begin{aligned} |E(x, y)| &\leq \frac{4.41e^{2.31}}{2}(0.1 + 0.1)^2 \\ &\leq (0.02)(4.41)e^{2.31} \\ &\leq 0.0882e^{2.31} \end{aligned}$$