1. (28 pts) Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Be sure to fully justify your answers.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)!}{(n!)^2}$ (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3^{2n}}{(n+1)^n}$ (c) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^{5/2}}$

Solution:

(a) First simplify the series.

$$\sum_{n=0}^{\infty} \frac{(n+1)!}{(n!)^2} = \sum_{n=0}^{\infty} \frac{(n+1)\cdot n!}{(n!)^2} = \sum_{n=0}^{\infty} \frac{n+1}{n!}$$

Then by the Ratio Test

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{n+2}{(n+1)!}\cdot\frac{n!}{n+1} = \lim_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{1}{n+1}\\ &= \lim_{n\to\infty}\frac{n+2}{(n+1)^2} \overset{LH}{=} \lim_{n\to\infty}\frac{1}{2(n+1)} = 0 < 1.\end{aligned}$$

The series is absolutely convergent.

(b) Apply the Root Test to $\displaystyle\sum_{n-1}^{\infty} (-1)^n \frac{3^{2n}}{(n+1)^n} = \sum_{n=1}^{\infty}\left(\frac{-3^2}{n+1}\right)^n$.

$\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{3^2}{n+1} = 0 < 1$, so the series is $\boxed{\text{absolutely convergent}}$.

Alternate Solution: Apply the Ratio Test.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{3^{2n+2}}{(n+2)^{n+1}}\cdot\frac{(n+1)^n}{3^{2n}}\\ &= \lim_{n\to\infty}\frac{3^{2n+2}}{3^{2n}}\cdot\left(\frac{n+1}{n+2}\right)^n\cdot\frac{1}{n+2}\\ &= \lim_{n\to\infty} 3^2\cdot\left(1-\frac{1}{n+2}\right)^n\cdot\frac{1}{n+2}\\ &= \lim_{n\to\infty} 3^2\cdot\frac{1}{e}\cdot\frac{1}{n+2} = 0 < 1\end{aligned}$$

The series is absolutely convergent.

(c) By the Direct Comparison Test,

$$\begin{aligned}&0 < \ln n < n\\ &0 < \frac{\ln n}{n^{5/2}} < \frac{n}{n^{5/2}} = \frac{1}{n^{3/2}}\end{aligned}$$

Because $\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series $\left(p = \frac{3}{2} > 1\right)$, the given series is $\boxed{\text{absolutely convergent}}$.

Alternate Solution: Apply the Limit Comparison Test, comparing to the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is convergent ($p = 2 > 1$).

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\ln n}{n^{5/2}} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{\ln n}{n^{1/2}} \\ &\stackrel{LH}{=} \lim_{n\to\infty} \frac{1/n}{\frac{1}{2}n^{-1/2}} = \lim_{n\to\infty} \frac{2}{n^{1/2}} = 0 \end{aligned}$$

Therefore the given series is absolutely convergent.

2. (28 pts) Let $f(x) = \frac{5}{1 - \frac{x}{2}}$.

   (a) Find a power series representation for $f(x)$. Simplify your answer.
   (b) What is the radius of convergence of the power series?
   (c) Does the power series converge at the endpoints of the interval of convergence? Justify your answer by writing out the endpoint series and determining whether they are absolutely convergent, conditionally convergent, or divergent.
   (d) Find a power series representation for $x^3 f'(x)$. Simplify your answer.
   (e) What is the sum of the series found in part (d)? Simplify your answer.

Solution:

(a) To find a power series for $f(x)$, begin with the Maclaurin series for $1/(1-x)$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$f(x) = 5 \cdot \frac{1}{1 - \frac{x}{2}} = 5 \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \boxed{\sum_{n=0}^{\infty} \frac{5}{2^n} x^n}$$

(b) The Maclaurin series for $1/(1-x)$ has a radius of 1, so the $f(x)$ series converges for $\left|\frac{x}{2}\right| < 1 \implies |x| < 2$. The radius of convergence is $\boxed{R = 2}$.

(c) The endpoints of the interval $|x| < 2$ are at $x = \pm 2$.

At $x = 2$, the series is $\sum_{n=0}^{\infty} \frac{5}{2^n} 2^n = \sum_{n=0}^{\infty} 5$, which diverges by the Test for Divergence because $\lim_{n\to\infty} 5 = 5 \neq 0$.

At $x = -2$, the series is $\sum_{n=0}^{\infty} \frac{5}{2^n}(-2)^n = \sum_{n=0}^{\infty} (-1)^n 5$, which diverges by the Test for Divergence because $\lim_{n\to\infty} (-1)^n 5$ does not exist.

Therefore the interval of convergence $(-2, 2)$ $\boxed{\text{does not}}$ include the endpoints.

(d)

$$\begin{aligned} x^3 f'(x) &= x^3 \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{5}{2^n} x^n\right) \\ &= x^3 \sum_{n=0}^{\infty} \frac{5}{2^n} n x^{n-1} \\ &= \boxed{\sum_{n=0}^{\infty} \frac{5}{2^n} n x^{n+2}} = \sum_{n=1}^{\infty} \frac{5}{2^n} n x^{n+2} \end{aligned}$$

(e) The sum of the series is

$$x^3 f'(x) = x^3 \frac{d}{dx}\left(\frac{5}{1-\frac{x}{2}}\right) = x^3\left(\frac{-5}{\left(1-\frac{x}{2}\right)^2}\right)\left(-\frac{1}{2}\right) = \frac{5x^3}{2\left(1-\frac{x}{2}\right)^2} = \boxed{\frac{10x^3}{(2-x)^2}}.$$

3. (20 pts) The function $g(x)$ has the Maclaurin series representation $\displaystyle\sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{2}{3^n} x^{n+2}$.

(For this problem, your answers should <u>fully simplify</u> all binomial coefficients.)

(a) Approximate the value of $g(1)$ using $T_3(x)$, the 3rd order Taylor polynomial for $g(x)$. Simplify your answer.

(b) Use the Alternating Series Estimation Theorem to find an error bound for the approximation found in part (a). You may assume that the conditions of the theorem are satisfied.

(c) Find a closed form (non-series) expression for $g(x)$.

Solution:

(a) The binomial coefficients for $n = 0$ and 1 evaluate to $\binom{-1/2}{0} = 1$ and $\binom{-1/2}{1} = -\frac{1}{2}$.

The third order Taylor polynomial is

$$T_3(x) = \binom{-1/2}{0} 2x^2 + \binom{-1/2}{1} \frac{2}{3} x^3 = 2x^2 - \frac{1}{3}x^3.$$

An approximation for $g(1)$ is $T_3(1) = 2 - \frac{1}{3} = \boxed{\frac{5}{3}}$.

(b) The series $\displaystyle\sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{2}{3^n}$ is alternating and satisfies the conditions of the Alternating Series Estimation Theorem. Let S equal the sum of the series and let $b_n = \left|\binom{-1/2}{n}\frac{2}{3^n}\right|$. Then $T_3(1)$ equals the partial sum $s_1 = b_0 - b_1$ and an error bound for $|R_1| = |S - s_1|$ is

$$b_2 = \binom{-1/2}{2} \frac{2}{3^2} = \frac{3}{8} \cdot \frac{2}{9} = \boxed{\frac{1}{12}}$$

because the binomial coefficient $\binom{-1/2}{2}$ equals $\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \cdot \frac{1}{2} = \frac{3}{8}$.

(c) Given the Maclaurin series

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n,$$

the function $g(x)$ equals

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{2}{3^n} x^{n+2} &= 2x^2 \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\frac{x}{3}\right)^n \\ &= 2x^2 \left(1+\frac{x}{3}\right)^{-1/2} = \boxed{\frac{2\sqrt{3}\,x^2}{\sqrt{3+x}}}.\end{aligned}$$

4. (12 pts) The following two problems are not related.

(a) Does the sum of the series exist? If so, find the sum. If not, explain why not.

$$\frac{10^1}{0!} - \frac{10^3}{2!} + \frac{10^5}{4!} - \frac{10^7}{6!} + \cdots$$

(b) The function $h(x)$ has the Taylor series representation $\displaystyle\sum_{n=1}^{\infty} \frac{n^3(x-3)^{n-1}}{2^{n+1}}$. Find the value of the 9th derivative of $h(x)$ evaluated at $x=3$. Leave your answer unsimplified.

Solution:

(a) Begin with the Maclaurin series for $\cos(10)$.

$$\begin{aligned}\cos x &= \frac{1}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ \cos(10) &= \frac{1}{0!} - \frac{10^2}{2!} + \frac{10^4}{4!} - \frac{10^6}{6!} + \cdots\end{aligned}$$

Factoring out 10 from the given series gives

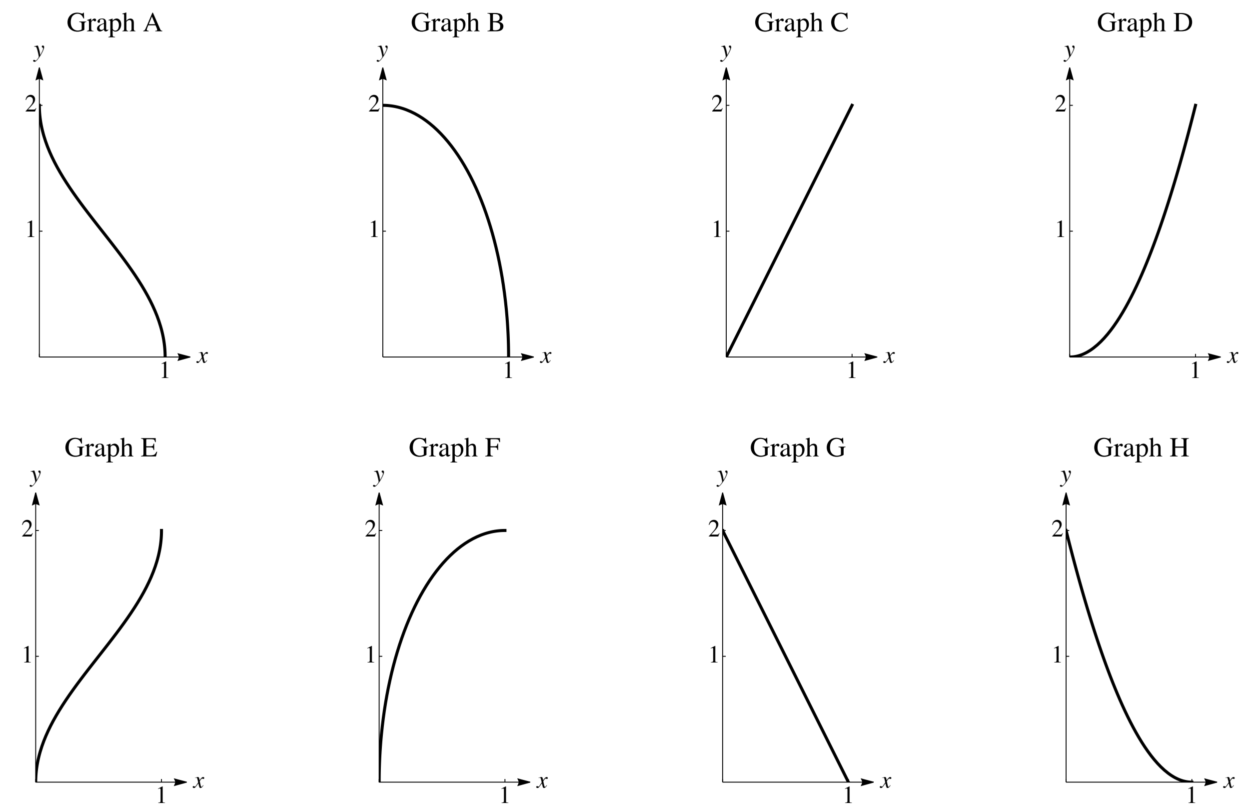
$$\begin{aligned}\frac{10^1}{0!} - \frac{10^3}{2!} + \frac{10^5}{4!} - \frac{10^7}{6!} + \cdots &= 10\left(\frac{1}{0!} - \frac{10^2}{2!} + \frac{10^4}{4!} - \frac{10^6}{6!} + \cdots\right) \\ &= \boxed{10\cos(10)}.\end{aligned}$$

(b) Match the x^9 term in the Taylor Series formula to the x^9 term in the $h(x)$ series. Let $n=10$.

$$\begin{aligned}\frac{h^{(9)}(3)}{9!}(x-3)^9 &= \frac{10^3}{2^{11}}(x-3)^9 \\ \frac{h^{(9)}(3)}{9!} &= \frac{10^3}{2^{11}} \\ h^{(9)}(3) &= \boxed{\frac{10^3\, 9!}{2^{11}}}.\end{aligned}$$

5. (12 pts) Match each pair of parametric equations to one of the graphs shown below.
No explanation is required.

(a)	$x = 1 - \ln t$	$y = 2(\ln t)^2$	$1 \le t \le e$	Graph ____
(b)	$x = \sin t$	$y = 2\cos t$	$0 \le t \le \frac{\pi}{2}$	Graph ____
(c)	$x = \cos^2 t$	$y = 2\sin^2 t$	$0 \le t \le \frac{\pi}{2}$	Graph ____
(d)	$x = -1 + \sec^2 t$	$y = 2\tan^2 t$	$0 \le t \le \frac{\pi}{4}$	Graph ____



Solution:

(a) Graph $\boxed{\text{H}}$. Replacing $\ln t$ with $1 - x$ gives $y = 2(1-x)^2$, which is a parabola with vertex at $(1, 0)$.

(b) Graph $\boxed{\text{B}}$. The equations $x = \sin t$, $y = \cos t$ correspond to a unit circle centered at the origin. In the given equations, y is scaled by a factor of 2, producing an ellipse with height equal to twice the width.

(c) Graph $\boxed{\text{G}}$. Substituting into the identity $\cos^2 t + \sin^2 t = 1$ gives $x + \frac{y}{2} = 1 \implies y = 2(1-x)$, which is a line with slope -2 and y-intercept 2.

(d) Graph $\boxed{\text{C}}$. Substituting into the identity $\sec^2 t + \tan^2 t = 1$ gives $(x+1) + \frac{y}{2} = 1 \implies y = 2x$, which is a line through the origin with slope 2.